K-THEORY AND NONCOMMUTATIVE GEOMETRY

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Résumé du cours N-COMPLEXES

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After a brief survey of some aspects of the (homological) BRS methods in physics, we introduce the basic notions on N-complexes. We describe the Kapranov monoidal structure for N-complexes and we explain in this framework our joint work with Richard Kerner on the corresponding generalization of graded differential algebras.

We then describe our work with Marc Henneaux on the N-complexes of tensor fields of mixed Young symmetry type which generalize the complex of differential forms on $\mathbb{R}^n$ and we explain the corresponding generalization of the Poincaré lemma. We give several applications of the latter in theoretical physics and in differential geometry.

We introduce the family of N-complexes associated with simplicial modules at root of the unit and explain how they compute the homology of these simplicial modules. We give several applications and in particular a physically inspirated one developed in collaboration with Ivan Todorov which relies to spectral sequences methods for N-complexes.
We move to our joint work with Roland Berger and Marc Wambst on the homogeneous algebras and we explain in details why the N-complex generalization of the Koszul complexes of quadratic algebras is conceptually involved and practically unavoidable here.
We then describe our work with Alain Connes on the cubic Yang-Mills algebra and discuss some higher degree generalizations.

$\boxed{\text{I - PHOTONS}}$

Ref : [22].

Ref : [1], [4].

$\boxed{\text{Photon 1-particle space}}$

$\mathbf{C}_+ = \{p | g^{\mu\nu} p_\mu p_\nu = p_0^2 - \vec{p}^2 = 0, \;\; p_0 > 0\}$

$$d\mu_0(p) = \left(\frac{1}{2\pi}\right)^3 \frac{d^3\vec{p}}{2p_0}, \;\; \mathcal{H} = \int_{C_+}^{\oplus} d\mu_0(p) \mathcal{H}_p$$

$\mathcal{H}_p = \mathcal{Z}_p / \mathcal{B}_p$ 2-dimensional Hilbert space

$$\mathcal{Z}_p = \{A_\mu \in \mathbb{C}^4 | p^\mu A_\mu = 0\} \subset \mathcal{C}_p = \mathbb{C}^4$$

$$\mathcal{B}_p = \{p_\mu \varphi | \varphi \in \mathbb{C}\} \subset \mathcal{Z}_p$$

Indefinite scalar product of $\mathcal{C}_p$

$$\langle A | A' \rangle = -g^{\mu\nu} \bar{A}_\mu A'_\nu$$

Positive on $\mathcal{Z}_p$ with isotropic $\mathcal{B}_p$

$\Rightarrow$ induces a Hilbert structure on $\mathcal{H}_p$

In fact $\mathcal{B}_p \subset \mathcal{B}_p^\perp = \mathcal{Z}_p$ in $\mathcal{C}_p$

$\boxed{\mathcal{H} \text{ as a homology}}$

Define

$$Q_p = Q : \mathcal{C}_p \to \mathcal{C}_p$$

$$Q(A)_\mu = p_\mu p^\nu A_\nu \Rightarrow < A|QA' > = < QA|A' >$$

and $Q^2 = 0$

$(\mathcal{C}_p, Q_p)$ differential vector space with homology $H(\mathcal{C}_p) = \mathcal{H}_p$

$\Rightarrow \mathcal{H}$ as the homology of $\mathcal{C}$

$\to$ OK at 1-particle level

Triplet $(\mathcal{C}_p, \mathcal{Z}_p, \mathcal{B}_p)$ with indefinite $< | >$ in connection with indecomposable group represent.

$\boxed{\text{1-particle complex (Ghosts)}}$

$$C_p = C_p^{-1} \oplus C_p^0 \oplus C_p^{+1}$$

$$C_p^0 = \mathcal{C}_p = \mathbb{C}^4, \quad C_p^{-1} = \mathbb{C}, \quad , C_p^{+1} = \mathbb{C}$$

ε^μ (real) canonical basis of $\mathbb{C}^4$
$\omega^{(\pm)}$ basis of $C_p^\pm$

$$\delta_p : C_p^n \to C_p^{n+1}, \quad \delta_p^2 = 0$$

$$\delta_p \omega^{(+)} = 0, \;\; \delta_p \varepsilon^\mu = \alpha p^\mu \omega^{(+)}, \quad \delta_p \omega^{(-)} = p_\mu \varepsilon^\mu$$

$$(\alpha \in \mathbb{C} \backslash \{0\})$$

$$\Rightarrow H(C_p) = H^0(C_p) = \mathcal{H}_p$$

$\Rightarrow$ another description.

In coordinates $c\omega^{(-)} + A_\mu \varepsilon^\mu + \tilde{c}\omega^{(+)}$

$$\delta c = 0, \;\; \delta A_\mu = p_\mu c, \;\; \delta \tilde{c} = \alpha p^\lambda A_\lambda$$

$\boxed{\text{Hermitian scalar product}}$

One defines $\langle\,|\,\rangle$ on C_p by setting

$$\langle \omega^{(-)} | \omega^{(-)} \rangle = 0, \quad \langle \omega^{(-)} | \varepsilon^{\mu} \rangle = 0,$$

$$\langle \omega^{(-)} | \omega^{(+)} \rangle = -\alpha^{-1}, \quad \langle \varepsilon^{\mu} | \varepsilon^{\nu} \rangle = -g^{\mu\nu},$$

$$\langle \varepsilon^{\mu} | \omega^{(+)} \rangle = 0, \quad \langle \omega^{(+)} | \omega^{(+)} \rangle = 0$$

δ_p is then hermitian.

Now one can take tensor products

$\Rightarrow$ natural necessity of ghosts (= graduation)

II - CONSTRAINTS

Ref : [22], [18].

Ref : [30], [38], [10], [56], [54], [29], [39], [57].

$\boxed{\text{Reduced phase space}}$

$V \subset M$ symplectic $\qquad \omega =$ symplectic form

$\omega_V = i^*\omega,\ \mathcal{E}(V) = \{X \in T(V) | i_X \omega_V = 0\}$
$d\omega_V = 0 \Rightarrow \mathcal{E}(V)$ involutive $\Rightarrow$ Foliation $\mathcal{F}$ of V

$M_0 = V/\mathcal{F}$, $\omega_0 = \text{proj}(\omega_V)$
(M_0, ω_0)= reduced phase space

Construction of $C^\infty(M_0)$ (=observables) by algebraic homological methods $\Rightarrow$ 2 stages

1. restriction from M to V
2. passage from V to $V/\mathcal{F}$

$\boxed{\text{Koszul resolution of } C^\infty(V)}$

$V \subset M$ closed, $I(V) = \{f \in C^\infty(M) | f \restriction V = 0\}$
(R_0)

$$\begin{cases} \text{I(V) generated by } u_\alpha \in C^\infty(M) \;\; \alpha \in \{1, \dots, m\} \\ \text{such that } du_1 \wedge \cdots \wedge du_m(x) \neq 0 \;\; \forall x \in V \end{cases}$$

$E = \mathbb{R}^m \otimes C^\infty(M)$ free $C^\infty(M)$-module with canonical basis π_α, $\alpha \in \{1, \dots, m\}$

$u \in E^*$ defined by $u(\pi_\alpha) = u_\alpha$ extends uniquely as graded derivation d_u of $\wedge\mathbb{R}^m \otimes C^\infty(M)$ which is $C^\infty(M)$-linear and one has $d_u^2 = 0$
$\Rightarrow$ complex $K(u) = (\wedge\mathbb{R}^m \otimes C^\infty(M), d_u)$ of free $C^\infty(M)$-modules

LEMMA 1 $H^n(K(u)) = 0$ *for* $n \geq 1$ *and*
$H^0(K(u)) = C^\infty(M)/I(V) = C^\infty(V)$

i.e. $K(u) \to C^\infty(V) \to 0$ is a free resolution of the $C^\infty(M)$-module $C^\infty(V)$

$\boxed{\text{Longitudinal forms}}$

V equipped with foliation $\mathcal{F}$
$(\wedge\mathcal{F})^{\perp}$ =$\{$forms vanishing on $\mathcal{F}\}$ is a differential ideal $\Rightarrow \Omega(V,\mathcal{F}) = \Omega(V)/(\wedge\mathcal{F})^{\perp}$ graded differential algebra of longitudinal forms
$\Omega(V,\mathcal{F})$ is a $C^{\infty}(V)$-module
$H^0(V,\mathcal{F}) \simeq C^{\infty}(V/\mathcal{F})$

(R_1) $\mathcal{F}$ is a free $C^{\infty}(V)$-module of rank m'

$(R_1) \Rightarrow C^{\infty}(V) \otimes \wedge\mathbb{R}^{m'} \simeq \Omega(V,\mathcal{F})$
$\xi_{\alpha'}$ basis of $\mathcal{F}$, $\theta^{\alpha'}$ dual basis identifies to the basis of $\mathbb{R}^{m'}$. The $\theta^{\alpha'}$ generate $\Omega(V,\mathcal{F})$ and

$$\begin{cases} d_{\mathcal{F}} f = \xi_{\alpha'}(f)\theta^{\alpha'}, \;\; f \in C^{\infty}(V) \\ \\ d_{\mathcal{F}}\theta^{\alpha'} = -\frac{1}{2} C^{\alpha'}_{\beta'\gamma'}\theta^{\beta'}\theta^{\gamma'} \end{cases}$$

with $[\xi_{\beta'},\xi_{\gamma'}] = C^{\alpha'}_{\beta'\gamma'}\xi_{\alpha'}$

$\boxed{\text{Subquotient}}$

$M \supset V$, $\mathcal{F}$ foliation of V with (R_0) and (R_1)

$\mathcal{K} = \oplus_{i,j} \mathcal{K}^{i,j}$ with

$\mathcal{K}^{i,j} = \wedge^{-i}\mathbb{R}^m \otimes C^\infty(M) \otimes \wedge^j \mathbb{R}^{m'}$, for $i \leq 0 \leq j$ and $\mathcal{K}^{i,j} = 0$ otherwise.

$\pi_\alpha, \theta^{\alpha'}$ basis of $\mathbb{R}^m$ and $\mathbb{R}^{m'}$
δ_0 = unique antiderivation with $\delta_0 \pi_\alpha = u_\alpha$, $\delta_0 \theta^{\alpha'} = 0$ and $\delta_0 f = 0$ for $f \in C^\infty(M)$

Lemma 1 $\Rightarrow H(\delta_0) = C^\infty(V) \otimes \wedge \mathbb{R}^{m'} = \Omega(V, \mathcal{F})$

LEMMA 2 *There are antiderivations δ_r of bidegree $(1-r, r)$ for $r \geq 1$ such that*
$\sum_{r+s=n} \delta_r \delta_s = 0$, $\forall n \in \mathbb{N}$
and such that δ_1 induces $d_{\mathcal{F}}$ on $H(\delta_0) = \Omega(V, \mathcal{F})$

Let us extend $\xi_{\alpha'}$ and $C^{\alpha'}_{\beta'\gamma'}$ as $\tilde{\xi}_{\alpha'}, \tilde{C}^{\alpha'}_{\beta'\gamma'} \in C^\infty(M)$ and set

$$\begin{cases} \delta_1 f = \tilde{\xi}_{\alpha'}(f)\theta^{\alpha'}, \quad \forall f \in C^\infty(M) \\ \\ \delta_1 \theta^{\alpha'} = -\frac{1}{2}\tilde{C}^{\alpha'}_{\beta'\gamma'}\theta^{\beta'}\theta^{\gamma'} \end{cases}$$

$(\delta_0\delta_1 + \delta_1\delta_0)f = \delta_0\delta_1 f = 0$,
$(\delta_0\delta_1 + \delta_1\delta_0)\theta^{\alpha'} = \delta_0\delta_1\theta^{\alpha'} = 0$

$\delta_1\delta_0\pi_\alpha = \delta_1 u_\alpha = \tilde{\xi}_{\alpha'}(u_\alpha)\theta^{\alpha'}$ which $= 0$ on V
$\Rightarrow \delta_1\delta_0\pi_\alpha = A^\beta_{\alpha'\alpha} u_\beta \theta^{\alpha'}$ for some $A^\beta_{\alpha'\alpha} \in C^\infty(M)$ setting

$$\delta_1 \pi_\alpha = -A^\beta_{\alpha'\alpha}\pi_\beta\theta^\alpha$$

one has $\delta_1\delta_0 + \delta_0\delta_1 = 0$ on the generators and the corresponding antiderivation coincides with $d_{\mathcal{F}}$ one $H(\delta_0)$.

The rest of the proof follows by induction on n using

$$H^{1-r,r+1}(\delta_0) = 0 = H^{1-r,r+2}(\delta_0)$$

for $r \geq 2$

Notice that $\delta_r = 0$ for $r > m'$ or $r > m+1$

THEOREM 1 $\delta = \sum_{r\geq 0} \delta_r$ *is a differential of* $\mathcal{K}$ *and* $H(\delta) = H(\mathcal{K}, \delta) \simeq H(V, \mathcal{F})$

$\mathcal{K} = \oplus_n \mathcal{K}^n$ ($\mathcal{K}^n = \oplus_{i+j=n} K^{i,j}$) is a graded algebra and $(\mathcal{K}, \delta)$ is a graded differential algebra.

$\boxed{\text{Super phase space}}$

(M, ω), $\;V \subset M \Rightarrow \mathcal{F}$

Assume (R_0) and (R_1)
$m \geq m'$, $m + m' = \dim(M) - \dim(M_0) = 2p$

Coisotropic case $\{I(V), I(V)\} \subset I(V)$
i.e. first class constraints
$\Rightarrow m = m'$ and $(R_0) \Rightarrow (R_1)$

Extending the Poisson bracket to $\mathcal{K}$ via
$\{\pi_\alpha, \theta^\beta\} = \delta_\alpha^\beta, \{\pi_\alpha, \pi_\beta\} = \{\theta^\alpha, \theta^\beta\} = 0$

$\{\pi_\alpha, f\} = \{\theta^\alpha, f\} = 0$
as super Poisson bracket.

$\Rightarrow \mathcal{K} =$ "functions" on a superphase space

$\delta\varphi = \{Q, \varphi\}$, $\;\forall \varphi \in \mathcal{K}$ for $Q = \delta(\pi_\alpha \theta^\alpha) \in \mathcal{K}^1$

Arbitrariness of the whole construction = canonical transf. of the super phase space.

$\boxed{\text{Appendix}}$

(M, ω) symplectic, $V \subset M$
$I(V) = \{f \in C^\infty(M) | f \restriction V = 0\}$
$I_1(V) = \{f \in C^\infty(M) | \{f, I(V)\} \subset I(V)\}$

LEMMA 3 *One has $C^\infty(V) = C^\infty(M)/I(V)$ and $I_1(V)$ is an ideal of $C^\infty(M)$ stable by $\{\bullet, \bullet\}$*

Notice that $(I(V))^2 \subset I_1(V)$ and that
supp $(f) \bigcap V = 0 \Rightarrow f \in I_1(V)$

LEMMA 4 *One has $\mathcal{F} =$ Ham$(I_1(V)) \restriction V$ (in $\Gamma T(M) \restriction V$) and the corresponding $I_1(V) \to \mathcal{F}$ is a homomorphism of Lie algebras and of $C^\infty(M)$-modules.*

$\boxed{\text{Complement : van Hove phenomenon}}$

Impossibility of a canonically-invariant quantization

THEOREM 2 *P = Poisson alg. of complex polynom. functions on $\mathbb{R}^{2n}$ (coordin. p_μ, q^ν); $\mathcal{A}$ = unital associative $\mathbb{C}$-algebra.*
$Q : P \to \mathcal{A}$, $\mathbb{C}$-linear such that $Q(1) = \mathbb{1}$ and $Q(\{f,g\}) = \frac{i}{\hbar}[Q(f), Q(g)], \forall f, g \in P$ ($\hbar \in \mathbb{C}\backslash\{0\}$).
Then the commutant of the $Q(p_\mu)$ and the $Q(q^\nu)$ in $\mathcal{A}$ is a noncommutative subalgebra

$Z(X,Y) \in \mathcal{A}[X^\mu; Y_\nu]$ defined by
$Z(X,Y) = \exp(-iQ(pX - qY))Q(\exp(i(pX - qY)))$
$\Rightarrow [Q(p_\mu), Z(X,Y)] = [Q(q^\nu), Z(X,Y)] = 0$
But
$e^{-i\frac{\hbar}{2}(XY' - X'Y)} Z(X,Y)Z(X',Y') - ((X,Y) \leftrightarrows (X',Y'))$
$= -i\hbar(XY' - X'Y)Z(X + X', Y + Y')$
$\Rightarrow [Z(X,Y), Z(X',Y')] \neq 0 \quad (\hbar \neq 0).$

$\boxed{\text{III - } N\text{-DIFFERENTIALS}}$

Ref : [22], [20], [25], [19], [21].

Ref : [51], [52], [40], [43], [61].

Note : Lemma 3 of this section uses the notion of q-numbers and the assumption (A_1) which are defined in the next section.

$\boxed{N\text{-differentials}}$

$E = \mathbb{K}$-module, $d \in \mathrm{End}(E)$ is a N-differential if $d^N = 0$; (E, d) is a N-differential module
$\Rightarrow$ Generalization of homology

$$_pH(E) = H_{(p)}(E) = \mathrm{Ker}(d^p)/\mathrm{Im}(d^{N-p})$$

$$p \in \{1, \dots, N-1\}$$

$H_{(p)}(E)$ (resp. $H_{(N-p)}(E)$) is the homology in degree 0 (resp. 1) of the $\mathbb{Z}_2$-complex

$$\begin{array}{ccccc} E & \stackrel{d^p}{\longrightarrow} & E & \stackrel{d^{N-p}}{\longrightarrow} & E \\ \| & & \| & & \| \\ E_0 & & E_1 & & E_0 \end{array}$$

More generally, N-differential object in an abelian category $\mathcal{C}$
$(E, d), E \in Ob(\mathcal{C}), d \in \mathrm{Hom}_{\mathcal{C}}(E, E)$ with $d^N = 0$.

$\boxed{\text{First examples}}$

$(E', d') \in N'$-diff.mod, $(E'', d'') \in N''$-diff.mod $\mapsto (E' \otimes E'', d' \otimes I'' + I' \otimes d'') \in (N' + N'' - 1)$-diff.mod.
$\Rightarrow$ construction of N-diff.mod. from $(N-1)$ ordinary differential modules (E_i, d_i)

$$E = \otimes_{i=1}^{N-1} E_i, \quad d = \sum_{i=1}^{N-1} I^{\otimes^{i-1}} \otimes d_i \otimes I^{\otimes^{N-i-1}}$$

$\mathbb{K}$ a field, $(E, d) \in N$-diff. vector space with $\dim(E) < \infty$. Decomposing E into indecomposable factors for $d \Rightarrow E \simeq \oplus_{n=1}^{N} \mathbb{K}^n \otimes \mathbb{K}^{m_n}$,

$$d \simeq \oplus_{n=2}^{N} D_n \otimes I_{m_n} \text{ with } D_n = \begin{pmatrix} \underline{0}_{n-1} & I_{n-1} \\ 0 & \underline{0}^t_{n-1} \end{pmatrix}$$

PROPOSITION 1 *One has for* $1 \leq k \leq N/2$

$$\dim H_{(k)}(E) = \dim H_{(N-k)}(E) = \sum_{j=1}^{k} \sum_{i=j}^{N-j} m_i$$

A basic lemma

(E, d) N-differential module
$Z_{(n)} = \mathrm{Ker}(d^n)$, $B_{(n)} = \mathrm{Im}(d^{N-n})$,
$H_{(n)} = Z_{(n)}/B_{(n)} = H_{(n)}(E)$
$Z_{(n)} \subset Z_{(m+1)}$, $B_{(n)} \subset B_{(n+1)}$
$\Rightarrow [i] : H_{(n)} \to H_{(n+1)}$
$dZ_{(n+1)} \subset Z_{(n)}$, $dB_{(n+1)} \subset B_{(n)}$
$\Rightarrow [d] : H_{(n+1)} \to H_{(n)}$

LEMMA 1 *Let ℓ and m be integers with $\ell \geq 1$, $m \geq 1$ and $\ell + m \leq N - 1$. Then the following hexagon $(\mathcal{H}^{\ell,m})$ of homomorphisms*

$$
\begin{tikzcd}
 & H_{(\ell+m)}(E) \arrow[r, "{[d]^m}"] & H_{(\ell)}(E) \arrow[dr, "{[i]^{N-(\ell+m)}}"] & \\
H_{(m)}(E) \arrow[ur, "{[i]^\ell}"] & & & H_{(N-m)}(E) \arrow[dl, "{[d]^\ell}"] \\
 & H_{(N-\ell)}(E) \arrow[ul, "{[d]^{N-(\ell+m)}}"] & H_{(N-(\ell+m))}(E) \arrow[l, "{[i]^m}"] &
\end{tikzcd}
$$

is exact.

Connecting homomorphism

Abelian category of N-differential modules

PROPOSITION 2 *Let* $0 \to E \overset{\varphi}{\to} F \overset{\psi}{\to} G \to 0$ *a short exact sequence of* N*-diff. modules.* $\exists$ *homomorphisms* $\partial : H_{(m)}(G) \to H_{(N-m)}(E)$ *for* $m \in \{1, \ldots, N-1\}$ *such that the following hexagons* $(\mathcal{H}_n)$ *of homomorphisms*

$$\begin{tikzcd}
 & H_{(n)}(F) \arrow[r, "\psi_*"] & H_{(n)}(G) \arrow[dr, "\partial"] & \\
H_{(n)}(E) \arrow[ur, "\varphi_*"] & & & H_{(N-n)}(E) \arrow[dl, "\varphi_*"] \\
 & H_{(N-n)}(G) \arrow[ul, "\partial"] & H_{(N-n)}(F) \arrow[l, "\psi_*"] &
\end{tikzcd}$$

are exact, for $n \in \{1, \ldots, N-1\}$.

$E \mapsto \mathbb{Z}_2$-complex $E_{(n)} : E \overset{d^n}{\to} E \overset{d^{N-n}}{\to} E$

$\partial : \begin{cases} H_{(n)}(G) \to H_{(N-n)}(E) \\ H_{(N-n)}(G) \to H_{(n)}(E) \end{cases}$ is the connecting homomorphism of the corresponding short exact sequence of $\mathbb{Z}_2$-complexes
$0 \to E_{(n)} \to F_{(n)} \to G_{(n)} \to 0$

$\boxed{\text{Homotopy}}$

E, F are N-differential modules
$\lambda, \mu : E \to F$ homomorphisms of N-diff. mod.
λ and μ are *homotopic* if there are module-homomorphisms $h_k : E \to F$ such that

$$\lambda - \mu = \sum_{k=0}^{N-1} d^{N-1-k} h_k d^k$$

LEMMA 2 *Let $\lambda, \mu : E \to F$ be homotopic. Then one has*

$$\lambda_* = \mu_* : H_{(n)}(E) \to H_{(n)}(F), \forall n \in \{1, \dots, N-1\}$$

COROLLARY 1 *Let E be a N-differential module such that there are module-endomorphisms $h_k : E \to E$ satisfying $\sum_{k=0}^{N-1} d^{N-1-k} h_k d^k = Id_E$. Then one has $H_{(n)}(E) = 0$, $\forall n \in \{1, \dots, N-1\}$.*

A useful acyclicity criterion

LEMMA 3 *Suppose that $\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1) and let E be a N-differential module such that there is a module-endomorphism h of E such that $hd - qdh = Id_E$. Then $H_{(n)}(E) = 0$, $\forall n \in \{1, \dots, N-1\}$.*

In fact in the unital associative $\mathbb{K}$-algebra $\mathcal{A}_q$ generated by H, D with relations $HD - qDH = \mathbb{1}$ one has

$$\sum_{k=0}^{N-1} D^{N-1-k} H^{N-1} D^k = [N-1]_q! \mathbb{1}$$

which implies the result.

To show this, it is sufficient to verify the identity in an appropriate homomorphic image of $\mathcal{A}_q$

$\boxed{\text{First applications of the basic lemma}}$

PROPOSITION 3 *Let $\varphi : E \to E'$ hom. of N-diff. mod. such that it induces isomorphisms*

$$\begin{aligned}\varphi_* : H_{(1)}(E) &\overset{\simeq}{\to} H_{(1)}(E'),\\ \varphi_* : H_{(N-1)}(E) &\overset{\simeq}{\to} H_{(N-1)}(E').\end{aligned}$$

Then $\varphi_* : H_{(n)}(E) \to H_{(n)}(E')$ is an isomorphism $\forall n \in \{1, \dots, N-1\}$.

Use $\mathcal{H}^{n,1}$ of Lemma 1 for $1 \leq n \leq N-1$ to obtain the result by induction on $n \geq 1$.

PROPOSITION 4 *Let E be a N-diff. mod. with $H_{(k)}(E) = 0$ for some $k \in \{1, \dots, N-1\}$. Then $H_{(n)}(E) = 0 \;\; \forall n \in \{1, \dots, N-1\}$.*

Use $\mathcal{H}^{1,k-1}$ to show $H_{(n)}(E) = 0$ for $1 \leq n \leq k$ and then $\mathcal{H}^{1,k}$ to show that $H_{(k+1)}(E) = 0$.

$\boxed{\text{IV - } N\text{-COMPLEXES}}$

Ref : [22], [20], [25], [19], [21].

Ref : [51], [52], [40], [43], [61].

$\boxed{N\text{-complexes}}$

$E = \oplus_{n\in\mathbb{Z}} E_n$ is $\mathbb{Z}$-graded with N-differential

d homogeneous of degree $\begin{cases} -1 & \text{chain } N\text{-complex} \\ \text{or} & \\ +1 & \text{cochain } N\text{-complex} \end{cases}$

(in the latter case the graduation is denoted in exponent i.e. $E = \oplus E^n$)

$\Rightarrow_p H(E) = H_{(p)}(E)$ is $\mathbb{Z}$-graded

$\mathbb{Z}_N$-complex if $E = \oplus_{n\in\mathbb{Z}_N} E_n$ is $\mathbb{Z}_N$-graded ($\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$) with d homogeneous of degree ∓ 1 (chain / cochain)

The ${}_pH(E) = H_{(p)}(E)$ are $\mathbb{Z}_N$-graded

$\mathbb{Z}_N$-$\underline{\text{compl}} \subset N$-$\underline{\text{compl}}$ (subcategory)

$\oplus_{[n]\in\mathbb{Z}_N} F_{[n]} \mapsto \oplus_{n\in\mathbb{N}} F_n$, $(F_n = F_{[n]}\ \forall n \in [n])$

N-$\underline{\text{compl}} \to \mathbb{Z}_N$-$\underline{\text{compl}}$ adjoint functor

$\oplus_{n\in\mathbb{N}} E_n \mapsto \oplus_{[n]\in\mathbb{Z}_N} E_{[n]}$, $E_{[n]} = \oplus_{n\in[n]} E_n$

$\boxed{q\text{-numbers}}$

$q \in \mathbb{K} \mapsto [\bullet]_q : \mathbb{N} \to \mathbb{K}$, $n \mapsto [n]_q$
$[0]_q = 0$, $[n]_q = \sum_{k=0}^{n-1} q^k$ for $n \geq 1$

Set $[n]_q! = \prod_{k=1}^{n} [k]_q$ and define
$\begin{bmatrix} n \\ m \end{bmatrix}_q$ for $0 \leq m \leq n$ by induction by setting
$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$ and for $0 \leq m \leq n-1$
$\begin{bmatrix} n \\ m \end{bmatrix}_q + q^{m+1} \begin{bmatrix} n \\ m+1 \end{bmatrix} = \begin{bmatrix} n+1 \\ m+1 \end{bmatrix}_q$

We shall make frequently use of the following assumptions for $\mathbb{K}$ and $q \in \mathbb{K}$, $N \in \mathbb{N}$ with $N \geq 2$
(A_0) $[N]_q = 0$
(A_1) $[N]_q = 0$ and $\exists [n]_q^{-1} \in \mathbb{K}$ for $1 \leq n \leq N-1$

$(A_0) \Rightarrow q^N = 1 \Rightarrow \exists q^{-1} \in \mathbb{K}$
(A_0) (resp. (A_1)) for $\mathbb{K}$ and $q \in \mathbb{K} \Leftrightarrow (A_0)$ (resp. (A_1)) for $\mathbb{K}$ and $q^{-1} \in \mathbb{K}$ (given N as above).

$E = \oplus_n E^n$ graded $\mathbb{K}$-module
F, G two $\mathbb{K}$-modules
$E \otimes F \to G$, $\alpha \otimes x \mapsto \alpha x$ $\mathbb{K}$-linear

Assume that E, F, G are equipped with $\mathbb{K}$-linear endomorphisms d with $d : E \to E$ homogeneous of degree 1 such that

(L) $d(\alpha x) = d(\alpha)x + q^a \alpha d(x)$; $\forall \alpha \in E^a$, $\forall x \in F$

Then one has by induction on n

$$d^n(\alpha x) = \sum_{m=0}^{n} q^{am} \left[\begin{array}{c} n \\ m \end{array} \right]_q d^{n-m}(\alpha) d^m(x)$$

In particular, if $q \in \mathbb{K}$ satisfies (A_1)
$\left[\begin{array}{c} N \\ m \end{array} \right]_q = 0$ for $1 < m < N$ and one has

$$d^N(\alpha x) = d^N(\alpha)x + \alpha d^N(x)$$

thus if (E, d) is a chain N-complex, (F, d) is a N-differential module and $G = E \otimes F$ then the right-hand side of (L) defines a N-differential on G.

$\boxed{\text{Examples}}$

1. $\mathbb{K} = \mathbb{Z}_N$ and $q = 1$
(A_0) satisfied
$(A_1) \Leftrightarrow N$ is a prime number.

2. $\mathbb{K} = \mathbb{C}$ and $q^N = 1$
$(A_0) \Leftrightarrow q \neq 1$
$(A_1) \Leftrightarrow q$ is a primitive N$^{\text{th}}$ root of 1.

$(E_n)_{n\in\mathbb{N}}$ presimplicial with faces
$\mathfrak{f}_i : E_n \to E_{n-1}, \quad i \in \{0, \dots, n\}$
$\mathfrak{f}_i\mathfrak{f}_j = \mathfrak{f}_{j-1}\mathfrak{f}_i$ if $i < j$

If $\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_0)
$d_0 = \sum_{k=0}^{n} q^k \mathfrak{f}_k : E_n \to E_{n-1}$

satisfies $d_0^N = 0$ on $E = \oplus_n E_n$
$\Rightarrow N$-complex (E, d_0)

1. $\to$ Mayer
2. $\to$ Kapranov
More general ansatz and computation $\to$ see later

$\boxed{\text{Matrices}}$

$\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1)

E^k_ℓ basis of $M_N(\mathbb{K})$; $(E^k_\ell)^i_j = \delta^k_j \delta^i_\ell$
$E^k_\ell E^r_s = \delta^k_s E^r_\ell$ and $\sum_{n=1}^N E^n_n = \mathbb{1}$

$\Rightarrow M_N(\mathbb{K})$ is $\mathbb{Z}_N$-graded by setting

$\deg(E^k_\ell) = k - \ell \quad \text{mod } (N)$

$e = \lambda_1 E^2_1 + \dots + \lambda_{N-1} E^N_{N-1} + \lambda_N E^1_N \in M_N(\mathbb{K})^1$
$d(A) = eA - q^a Ae, \ A \in M_N(\mathbb{K})^a$
$d^N = 0 \Rightarrow (M_N(\mathbb{K}), d)$ is a $\mathbb{Z}_N$-complex, in fact a $\mathbb{Z}_N$-graded q-differential algebra.

$e^N = \lambda_1 \dots \lambda_N \mathbb{1}$,
$e^{N-1} d(A) - q d(e^{N-1} A) = (1-q) \lambda_1 \dots \lambda_N A$

If $\exists (1-q)^{-1}, \lambda_i^{-1} \in \mathbb{K}$, then $H_{(n)}(M_N(\mathbb{K}), d) = 0$

$\boxed{\text{V - TENSOR PRODUCT AND } q\text{-DIFFERENTIAL ALGEBRAS}}$

Ref : [22], [20], [25], [21].

Ref : [50], [9].

$\boxed{\text{Monoidal categories}}$

$\mathcal{C}$ *monoidal* if

1) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ cov. funct. with natural isom.
$a_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ such that
$(a_{A,B,C} \otimes \mathrm{Id}_D) \circ a_{A,B\otimes C,D} \circ (\mathrm{Id}_A \otimes a_{B,C,D}) =$
$a_{A\otimes B,C,D} \circ a_{A,B,C\otimes D} :$
$A \otimes (B \otimes (C \otimes D)) \to ((A \otimes B) \otimes C) \otimes D$

2) $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ *unit object* with natural isom.
$\ell_A : \mathbb{1} \otimes A \to A$, $r_A : A \otimes \mathbb{1} \to A$ such that
$\mathrm{Id}_A \otimes r_B = r_{A\otimes B} \circ a_{A,B,\mathbb{1}} : A \otimes (B \otimes \mathbb{1}) \to A \otimes B$
$\mathrm{Id}_A \otimes \ell_B = (r_A \otimes \mathrm{Id}_B) \circ a_{A,\mathbb{1},B} : A \otimes (\mathbb{1} \otimes B) \to A \otimes B$
$\ell_{A\otimes B} = (\ell_A \otimes \mathrm{Id}_B) \circ a_{\mathbb{1},A,B} : \mathbb{1} \otimes (A \otimes B) \to A \otimes B$
$\mathcal{C}$ is monoidal *strict* if a, ℓ, r are identities.

Every monoidal category is equivalent to a strict one

$\boxed{\text{Monoids}}$

$\mathcal{C}$ monoidal
$A \in \mathrm{Ob}(\mathcal{C})$ is a $\mathcal{C}$-*monoid* if $\exists$ morphisms
$\mu : A \otimes A \to A$ and $\eta : \mathbb{1} \to A$ such that
$\mu \circ (\mathrm{Id}_A \otimes \mu) = \mu \circ (\mu \otimes \mathrm{Id}_A) \circ a_{A,A,A}$, (associativity)
$\mu \circ (\mathrm{Id}_A \otimes \eta) = r_A$ and $\mu \circ (\eta \otimes \mathrm{Id}_A) = \ell_A$
$\mu =$ "multiplication", $\eta =$ "unit" (of A)

$\mathbb{K}$-Mod, $\otimes_{\mathbb{K}}$, $\mathbb{1} = \mathbb{K}$ monoidal
A monoid in $\mathbb{K}$-Mod $\Leftrightarrow$ A associative $\mathbb{K}$-algebra

More generally if $\mathcal{C}$ is monoidal and if its objects are $\mathbb{K}$-modules, a monoid in $\mathcal{C}$ will be called and associative algebra of $\mathcal{C}$

$\boxed{\text{Examples}}$

1) $\mathcal{C}$ = cochain complexes of $\mathbb{K}$-modules

$$\begin{cases} (E \otimes F)^n = \oplus_{r+s=n} E^r \otimes F^s \\ d(e \otimes f) = d(e) \otimes f + (-1)^{\mathsf{deg}(e)} e \otimes d(f) \end{cases}$$

A $\mathcal{C}$-monoid = A graded differential algebra

2) $\mathcal{C}$= cochain N-complexes of $\mathbb{K}$-modules
$\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1)

$$\begin{cases} (E \otimes F)^n = \oplus_{r+s=n} E^r \otimes F^s \\ d(e \otimes f) = d(e) \otimes f + q^{\mathsf{deg}(e)} e \otimes d(f) \end{cases}$$

A $\mathcal{C}$-monoid = A *graded q-differential algebra*

3) Under (A_1) for $\mathbb{K}$ and $q \in \mathbb{K}$ the above formulae induce a monoidal structure for the category of $\mathbb{Z}_N$-complexes of $\mathbb{K}$-modules.
$\Rightarrow \mathbb{Z}_N$-graded q-differential algebras

$$\boxed{\text{Another approach: The Hopf algebra } \mathcal{D}_q}$$

$\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1)
$\mathcal{D}_q =$ associative unital $\mathbb{K}$-algebra generated by d and Γ with relations
$d^N = 0, \;\; \Gamma^N = \mathbb{1}, \;\; \Gamma d = q d \Gamma$
$\mathcal{D}_q$ is a Hopf algebra with

$$\begin{cases} \Delta(d) = d \otimes \mathbb{1} + \Gamma \otimes d, & \Delta\Gamma = \Gamma \otimes \Gamma \\ \varepsilon(d) = 0, & \varepsilon(\Gamma) = 1 \\ S(d) = -\Gamma^{N-1} d, & S(\Gamma) = \Gamma^{N-1} \end{cases}$$

$E = \oplus_{n \in \mathbb{Z}_N} E^n \qquad \mathbb{Z}_N$-complex

$$\begin{cases} d \mapsto d = N\text{-differential of } E \\ \Gamma \mapsto \text{ multiplication by } q^p \text{ on } E^p, \; \forall p \in \mathbb{Z}_N \end{cases}$$

$\Rightarrow E$ is a $\mathcal{D}_q$-module and the (q) tensor product of $\mathbb{Z}_N$-complexes is induced by the coproduct of $\mathcal{D}_q$

Case of N-complexes (i.e. $\mathbb{Z}$-graduation)

$\boxed{\text{Hochschild cochains I}}$

$\mathcal{A}$ unital associative $\mathbb{K}$-algebra, (A_1) for $q \in \mathbb{K}$
$\mathcal{M}, \mathcal{M}'$ $(\mathcal{A}, \mathcal{A})$-bimodules
$\omega \in C^n(\mathcal{A}, \mathcal{M}) \mapsto d_1\omega \in C^{n+1}(\mathcal{A}, \mathcal{M})$
$d_1(\omega)(x_0, \dots, x_n) = x_0\omega(x_1, \dots, x_n)$
$+ \sum_{k=1}^n q^k\omega(x_0, \dots, x_{k-1}x_k, \dots, x_n)$
$-q^n\omega(x_0, \dots, x_{n-1})x_n$

$d_1^N = 0, \quad d_1(\omega \cup \omega') = d_1(\omega) \cup \omega' + q^n\omega \cup d_1(\omega')$
for $\omega \in C^n(\mathcal{A}, \mathcal{M})$, $\omega' \in C(\mathcal{A}, \mathcal{M}')$

$\Rightarrow (C(\mathcal{A}, \mathcal{A}), d_1)$ is a graded q-differential algebra

$C(\mathcal{A}) = \oplus_n(\otimes^n\mathcal{A})^*$ is a graded algebra
$\mu_q^*(\omega)(x_0, \dots, x_n) =$
$= \sum_{k=1}^n q^{k-1}\omega(x_0, \dots, x_{k-1}x_k, \dots, x_n)$

$(C(\mathcal{A}), \mu_q^*)$ is a graded q-differential algebra

$\boxed{\text{The graded algebra } \mathfrak{T}(\mathcal{A})}$

$\mathfrak{T}(\mathcal{A}) = T_{\mathcal{A}}(\mathcal{A} \otimes \mathcal{A}) = \oplus_n (\otimes^{n+1} \mathcal{A})$

Generated by $\mathcal{A}$ in degree 0 and by the free generator $\tau = \mathbb{1} \otimes \mathbb{1}$ in degree 1
$x_0 \otimes \cdots \otimes x_n = x_0 \tau x_1 \dots \tau x_n \in \mathfrak{T}^n(\mathcal{A})$

PROPOSITION 1 *(Universal property) Let $\mathfrak{A} = \oplus_{n \geq 0} \mathfrak{A}^n$ be a graded $\mathbb{K}$-algebra. For any unital $\mathbb{K}$-algebra homomorphism $\varphi : \mathcal{A} \to \mathfrak{A}^0$ and for any $\alpha \in \mathfrak{A}^1$, $\exists !$ graded algebra homomorphism $\mathfrak{T}_{\varphi,\alpha} : \mathfrak{T}(\mathcal{A}) \to \mathfrak{A}$ which extends φ and sends τ on α.*

Take $\mathfrak{A} = C(\mathcal{A}, \mathcal{A}), \varphi = \mathrm{Id}_{\mathcal{A}}$ and
$\alpha \in \mathrm{Id}_{\mathcal{A}} \in C^1(\mathcal{A}, \mathcal{A}) \Rightarrow \Psi = \mathfrak{T}_{\varphi,\alpha}$ given by
$\Psi(x_0 \otimes \cdots \otimes x_n)(y_1, \dots, y_n) = x_0 y_1 x_1 \dots y_n x_n$

$\boxed{q\text{-differential structure on } \mathfrak{T}(\mathcal{A})}$

$\mathcal{A}$ unital assoc. alg. as above, (A_1) for $q \in \mathbb{K}$

PROPOSITION 2 $\exists!$ $d_1 : \mathfrak{T}(\mathcal{A}) \to \mathfrak{T}(\mathcal{A})$ *linear homogeneous of degree 1 satisfying the graded q-Leibniz rule*
$d_1(\alpha\beta) = d_1(\alpha)\beta + q^{|\alpha|}\alpha d_1(\beta)$
such that
$d_1(x) = \mathbb{1} \otimes x - x \otimes \mathbb{1} = \tau x - x\tau, \quad \forall x \in \mathcal{A}$
and
$d_1(\tau) = \tau^2$ *(i.e.* $d_1(\mathbb{1} \otimes \mathbb{1}) = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$*).*
Then one has $d_1^N = 0$*.*

In fact, by induction on n, one has
$d_1^n(x) = [n]_q!\tau^{n-1}d_1(x)$ and $d_1^n(\tau) = [n]_q!\tau^{n+1}$

$(\mathfrak{T}(\mathcal{A}), d_1)$ is a graded q-differential algebra and one verifies that
$\Psi : \mathfrak{T}(\mathcal{A}) \to C(\mathcal{A}, \mathcal{A})$
is a homomorphism for this structure i.e.
$\Psi \circ d_1 = d_1 \circ \Psi$

$\boxed{\text{Universal } q\text{-differential envelope}}$

$\mathcal{A}, \mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1) as above.
Let $\Omega_q(\mathcal{A})$ be the graded q-differential subalgebra of $(\mathfrak{T}(\mathcal{A}), d_1)$ generated by $\mathcal{A} = \mathfrak{T}^0(\mathcal{A})$ and let d denote the N-differential of $\Omega_q(\mathcal{A})$ (induced by d_1).

THEOREM 1 *(Universal property) Any homomorphism $\varphi : \mathcal{A} \to \mathfrak{A}^0$ of unital associative $\mathbb{K}$-algebras of $\mathcal{A}$ into the subalgebra of degree 0 elements of a graded q-differential algebra $\mathfrak{A}$ extends uniquely as a homomorphism*
$\Omega_q(\varphi) : \Omega_q(\mathcal{A}) \to \mathfrak{A}$:
of graded q-differential algebras.

This universal property characterizes $\Omega_q(\mathcal{A})$ uniquely up to an isomorphism

$\Omega_q(\mathrm{Id}_{\mathcal{A}}) : \Omega_q(\mathcal{A}) \to \mathfrak{T}(\mathcal{A})$ is the inclusion
$\Omega_q(\mathrm{Id}_{\mathcal{A}}) : \Omega_q(\mathcal{A}) \to C(\mathcal{A}, \mathcal{A})$ is induced by Ψ.

A direct construction of $\Omega_q(\mathcal{A})$ is possible.

Homological properties of $\Omega_q(\mathcal{A})$

PROPOSITION 3 *Assume there is a linear form ω on $\mathcal{A}$ such that $\omega(\mathbb{1}) = 1$. Then $H^n_{(k)}(\mathfrak{T}(\mathcal{A}), d_1) = H^n_{(k)}(\Omega_q(\mathcal{A})) = 0$ for $n \geq 1$ and $H^0_{(k)}(\mathfrak{T}(A), d_1) = H^0_{(k)}(\Omega_q(\mathcal{A})) = \mathbb{K}$ for any $k \in \{1, \dots, N-1\}$.*

Define the cochain N-complex (E, d) by $E = \mathbb{K}e_{-(N-1)} \oplus \cdots \oplus \mathbb{K}e_{-1} \oplus \mathfrak{T}(\mathcal{A})$ with $d = d_1$ on $\mathfrak{T}(\mathcal{A})$ and $de_{-1} = \mathbb{1}$, $de_{-i} = e_{-(i-1)}$ for $2 \leq i \leq N-1$.

Define $h : E \to E$ linear of degree -1 by
$h(x_0 \otimes \cdots \otimes x_n) = \omega(x_0) x_1 \otimes \cdots \otimes x_n, \quad n \geq 1$
$h(x_0) = -q^{-1}\omega(x_0)e_{-1}$ and
$h(e_{-i}) = -q^{-(i+1)}[i+1]_q e_{-(i+1)}, \ 1 \leq i \leq N-2$
$h(e_{(-N-1)}) = 0$
Then $hd - qdh = I$ which implies the result since $\mathbb{K}e_{-(N-1)} \oplus \cdots \oplus \mathbb{K}e_{-1} \oplus \Omega_q(\mathcal{A})$ is stable by d and h.

VI - N-COMPLEXES OF IRREDUCIBLE TENSOR FIELDS

Ref : [22], [23], [24].

Ref : [5].

$\boxed{\text{Notations}}$

$(x^\mu) = (x^1, \ldots, x^D)$ coord. in $\mathbb{R}^D$

$\partial_\mu = \partial/\partial x^\mu$ partial deriv.
(ident. flat torsion free connection of $\mathbb{R}^D$)

T = cov. tens. field of degree = p

$$x \mapsto T_{\mu_1 \ldots \mu_p}(x)$$

∂T = cov. tens. field of degree = $p + 1$

$$x \mapsto \partial_{\mu_{p+1}} T_{\mu_1 \ldots \mu_p}(x)$$

$T \mapsto \partial T$ is a first order differential operator, homogeneous of degree 1.
($\{$cov. tens. fields$\}$ = graded vector space)

$\boxed{\text{Differential forms}}$

$$\Omega(\mathbb{R}^D) = \bigoplus_{p\in\mathbb{N}} \Omega^p(\mathbb{R}^D)$$

$$\Omega^p(\mathbb{R}^D) = \{\text{antisym. cov. tens. fields of degree } p\}$$

$$d = (-1)^p A_{p+1} \circ \partial : \Omega^p(\mathbb{R}^D) \to \Omega^{p+1}(\mathbb{R}^D)$$

$$[\partial_\mu, \partial_\nu] = 0 \Rightarrow d^2 = 0$$

$$\Rightarrow \mathrm{Im}(d) \subset \mathrm{Ker}(d)$$

$$H(\Omega(\mathbb{R}^D)) = \frac{\mathrm{Ker}(d)}{\mathrm{Im}(d)} = \bigoplus_p H^p(\Omega(\mathbb{R}^D))$$

$$H^p = \{\omega \in \Omega^p | d\omega = 0\}/d\Omega^{p-1}$$

$$H^0(\Omega(\mathbb{R}^D)) = \{\text{constant functions}\} = \mathbb{C}\mathbb{1}$$

LEMMA 1 *(Poincaré lemma)*

$$H^p(\Omega(\mathbb{R}^D)) = 0, \quad \forall p \geq 1$$

Generalization

$\omega \in \Omega^p(\mathbb{R}^D)$, ω : specific case of fields of irreducible tensors of degree $= p$

$\longleftrightarrow$ Young diagram with 1 column $\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline \end{array}$

$$p \mapsto Y_p, \quad (Y) = (Y_p)_{p\in\mathbb{N}}$$

$$\Omega_{(Y)}(\mathbb{R}^D) = \bigoplus_p \Omega^p_{(Y)}(\mathbb{R}^D)$$

$$\Omega^p_{(Y)}(\mathbb{R}^D) = \{\text{cov. tens. fields in } \mathrm{Im}(Y_p)\}$$

$$d = (-1)^p \mathbf{Y}_{p+1} \circ \partial : \Omega^p_{(Y)}(\mathbb{R}^D) \to \Omega^{p+1}_{(Y)}(\mathbb{R}^D)$$

LEMMA 2 $N \in \mathbb{N}$ *with* $N \geq 2$; (Y) *such that* # *columns* $(Y_p) < N$ *(i.e.* $\leq N-1$*)* $\forall p \in \mathbb{N}$
$\Rightarrow d^N = 0$.

$$H_{(k)}(\Omega_{(Y)}(\mathbb{R}^D)) = \mathrm{Ker}(d^k)/\mathrm{Im}(d^{N-k})$$

$k \in \{1, \ldots, N-1\}$ (graded spaces)

$$\boxed{\Omega_N(\mathbb{R}^D)}$$

(Y) is chosen as satisfying the conditions of the lemma in a "maximal way" , i.e. one fills the lines of $N-1$ cells, etc. $\Rightarrow (Y^N) = (Y^N_p)_{p\in\mathbb{N}}$

$$\Rightarrow \Omega_N(\mathbb{R}^D) = \bigoplus_p \Omega^p_N(\mathbb{R}^D) = \Omega_{(Y^N)}(\mathbb{R}^D)$$

$$\text{with } d^N = 0$$

THEOREM 1 *(Generalized Poincaré lemma)*

$$H^{n(N-1)}_{(k)}(\Omega_N(\mathbb{R}^D)) = 0, \quad \forall n \geq 1, \quad \forall k.$$

One has $H^0_{(k)}$ = polynomials of degrees $< k$

<u>Remark</u> : The $H^{n(N-1)+p}_{(k)}(\Omega_N(\mathbb{R}^D))$ are infinite dim. for $n \geq 1$ and $1 \leq p < N-1$. (The $H^p_{(k)}$ are finite dim. for $0 \leq p \leq N-1$).

$\boxed{\text{Higher spin gauge fields}}$

Spin $= 1 \leftarrow$ (e.m.) $\rightarrow N = 2$

$$
\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^D) & \xrightarrow{d} & \Omega^1(\mathbb{R}^D) & \xrightarrow{d} & \Omega^2(\mathbb{R}^D) & \xrightarrow{d} & \Omega^3(\mathbb{R}^D) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{gauges} & & A_\mu & & F_{\mu\nu} & & \sim \text{ Bianchi}
\end{array}
$$

Spin $= 2 \leftarrow$ (grav.) $\rightarrow N = 3$

$$
\begin{array}{ccccccc}
\Omega^1_3(\mathbb{R}^D) & \xrightarrow{d} & \Omega^2_3(\mathbb{R}^D) & \xrightarrow{d^2} & \Omega^4_3(\mathbb{R}^D) & \xrightarrow{d} & \Omega^5_3(\mathbb{R}^D) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{gauges} & & h_{\mu\nu} & & R_{\lambda\rho,\mu\nu} & & \sim \text{ Bianchi}
\end{array}
$$

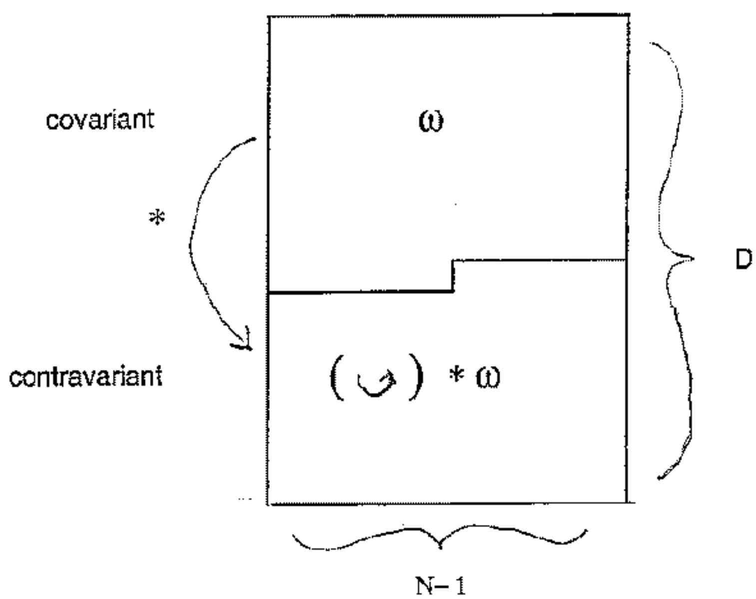
Spin $S \geq 1 \leftrightarrow N = S + 1$

$$\Omega^{S-1}_{S+1} \xrightarrow{d} \Omega^{S}_{S+1} \xrightarrow{d^S} \Omega^{2S}_{S+1} \xrightarrow{d} \Omega^{2S+1}_{S+1}$$

Other applications e.g. $S = 2$
$d(h) = 0 \leftrightarrow h = d^2\phi$ i.e. $h_{\mu\nu} = \partial_\mu \partial_\nu \phi$

$\boxed{\text{Duality}}$

Contracting columns by $\varepsilon^{\mu_1\cdots,\mu_D}$



$\Rightarrow$ Other applications. Example :

$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = T^{\nu\mu}$ ($\leftarrow \boxed{\mu \mid \nu}$)

$T^{\mu\nu} = \varepsilon^{\mu\mu_1\ldots\mu_{D-1}}\varepsilon^{\nu\nu_1\ldots\nu_{D-1}}\omega_{\mu_1\ldots\mu_{D-1},\nu_1\ldots\nu_{D-1}}$

$\partial_\mu T^{\mu\nu} = 0 \Leftrightarrow d\omega = 0$ in $\Omega_3(\mathbb{R}^D)$

$\Leftrightarrow \omega = d^2\rho \qquad \Leftrightarrow T^{\mu\nu} = \partial_\lambda\partial_\rho \qquad R^{\lambda\mu,\rho\nu}$

$\uparrow \qquad\qquad\qquad\qquad \uparrow$

(gen. Poincaré) $\qquad \left(\begin{array}{|c|c|}\hline \lambda & \rho \\ \hline \mu & \nu \\ \hline\end{array}\right)$

$\boxed{\text{Calculus on manifolds ?}}$

$\mathbb{R}^D \mapsto V$ manifold, $\dim(V) = D$
$\Omega_{(Y)}(V)$, $\Omega_N(V)$ well defined,
but not $T \mapsto \partial T$.

One must choose a linear connection ∇

$$\Rightarrow d_\nabla = (-1)^p \mathbf{Y}_{p+1} \circ \nabla : \Omega^p_{(Y)} \to \Omega^{p+1}_{(Y)}$$

$$\text{but } d^N_\nabla \neq 0$$

because of torsion and curvature of ∇
($\Rightarrow$ result at the level of symbol).

LEMMA 3 $(Y), N$ *as before.*
d^N_∇ *is of order* $N-1$
and if ∇ *is torsion free*
d^N_∇ *is of order* $N-2$*.*

$\boxed{\text{Computations for } N = 3}$

$H^0_{(1)} \simeq \mathbb{R}$, $H^0_{(2)} \simeq \mathbb{R} \oplus \mathbb{R}^D$
$H^1_{(1)} \simeq \{X | \partial_\mu X_\nu + \partial_\nu X_\mu = 0\} \simeq \mathbb{R}^D \oplus \wedge^2 \mathbb{R}^D$
$H^1_{(2)} \simeq \{X | \partial_\lambda(\partial_\mu X_\nu - \partial_\nu X_\mu) = 0\}/\{d\varphi\} \simeq \wedge^2 \mathbb{R}^D$

$\omega = 2\text{-form} \mapsto t = Y_3 \circ \partial\omega \in \Omega^3_3(\mathbb{R}^D)$
$t_{\mu\lambda\nu} = C^{\text{te}}(2\partial_\lambda \omega_{\mu\nu} + \partial_\mu \omega_{\lambda\nu} - \partial_\nu \omega_{\lambda\mu})$
$\Rightarrow dt \cong 0$ in $\Omega_3(\mathbb{R}^D)$

$t = dh$ i.e. $t_{\mu\lambda\nu} = \partial_\nu h_{\mu\lambda} - \partial_\mu h_{\nu\lambda} \Leftrightarrow$
$(*)\ \omega_{\mu\nu} = a_{\mu\nu\rho} x^\rho + \partial_\mu X_\nu - \partial_\nu X_\mu,\ a \in \wedge^3 \mathbb{R}^D$

and then $t = d^2 X$ in $\Omega_3(\mathbb{R}^D)$
$\Rightarrow H^3_{(1)}$ and $H^3_{(2)}$ contain the ∞-dim. space $\{$2-forms$\}$ / $\{$2-forms$(*)\}$.

Similar construction $\Rightarrow \dim(H^{2n+1}_{(k)}) = \infty$.
Basic lemma $+ H^{2n}_{(k)} = 0 \Rightarrow H^{2n+1}_{(1)} \simeq H^{2n+1}_{(2)}$
$(n \geq 1)$

$\boxed{\text{Computations for } N \geq 3}$

The construction for $N = 3$ generalizes for $N \geq 3$
$\Rightarrow \dim\left(H^m_{(k)}\right) = \infty$ for $m \geq N$ and $m \neq (N-1)p$
For $k + m \leq N - 1$, $H^m_{(k)} \simeq$

$$\{S | \sum_{\pi \in \mathfrak{S}_{k+m}} \partial_{\mu_{\pi(1)}} \dots \partial_{\mu_{\pi(k)}} S_{\mu_{\pi(k+1)} \dots \mu_{\pi(k+m)}} = 0\}$$

$\Rightarrow \dim H^m_{(k)} < \infty$ (polyn. degrees $< k + m$)
Basic lemma $+ H^{(N-1)p}_{(k)} = 0$ for $p \geq 1$
$\Rightarrow$ 4-terms exact sequences

$$0 \stackrel{[d]^k}{\rightarrow} H^{k-1}_{(\ell)} \stackrel{[i]^{N-k-\ell}}{\rightarrow} H^{k-1}_{(N-k)} \stackrel{[d]^\ell}{\rightarrow} H^{k+\ell-1}_{(N-k-\ell)} \stackrel{[i]^k}{\rightarrow}$$
$$\stackrel{[i]^k}{\rightarrow} H^{k+\ell-1}_{(N-\ell)} \stackrel{[d]^{N-k-\ell}}{\rightarrow} 0$$

for $1 \leq k, \ell, \quad k + \ell \leq N - 1$
$\Rightarrow \dim H^m_{(k)}$ $(< \infty)$ for $m < N - 1$ as functions of the $H^m_{(k)}$ for $k + m \leq N - 1$ $(k \geq 1)$

$\boxed{\text{Young diagrams}}$

Y= partition of $|Y| \in \mathbb{N} \leftrightarrow$ rows of lenghts m_i
$m_i \geq \cdots \geq m_r > 0, \;\; \sum m_i = |Y|$

(Drawing $\rightarrow$) columns of lengths $\tilde{m}_j$
$\tilde{m}_1 \geq \cdots \geq \tilde{m}_c > 0, \; \sum \tilde{m}_j = |Y|$
$m_1 = c, \;\; \tilde{m}_1 = r$

$\tilde{Y}$ = dual partition or diagram (see the drawing)

$Y' \subset Y$ inclusion clear
$Y' \subset\subset Y$ strong inclusion means
$m_1 \geq m'_1$ and $\tilde{m}_c \geq \tilde{m}'_1$

$\Rightarrow$ *contraction* $\mathcal{C}(Y|Y')$ for $Y' \subset\subset Y$ (see drawing)

$\boxed{\text{Schur modules}}$

E vect. sp., $\dim(E) = D < \infty$, E^*= dual
$\phi : E^{|Y|} \to F$ multilinear
(i) ϕ antisym. in entries of each column
(ii) antisym. in entries of a column with another entry of Y on the right-hand vanishes
Morphism $\phi \to \phi' = f \in \mathrm{Hom}(F, F')$ such that $\phi' = f \circ \phi$
Initial object = Schur module
$(\bullet)^Y : E^{|Y|} \to E^Y$

Construction $E^Y \subset E^{\otimes |Y|} \simeq$ mult. forms on $(E^*)^{|Y|}$
$T : (E^*)^{|Y|} \to \mathbb{R}$ satisfying (i) and (ii)
T arbitrary $\mapsto \mathcal{Y}(T) \in E^Y$
$\mathcal{Y}(T) = \sum_{p \in R} \sum_{q \in C} (-1)^{\varepsilon(q)} T \circ p \circ q$

$$\begin{cases} C \text{ permutes entries of each column} \\ R \text{ permutes entries of each row} \end{cases}$$

$\mathcal{Y}^2 = C^{\mathrm{te}} \mathcal{Y} \to \mathbf{Y}^2 = \mathbf{Y}$, *Young symmetrizer*

$\boxed{\text{Schur modules II}}$

$E^{\otimes^n} = E^{\otimes^n} \otimes_{\mathfrak{S}_n} \mathbb{K}(\mathfrak{S}_n)$

$\pi^{\otimes^n}(GL(E))' \simeq \{\mathrm{Im}(\mathfrak{S}_n) \text{ in } E^{\otimes^n}\}''$

$|Y| = n, E^Y = E^{\otimes^n} \otimes_{\mathfrak{S}_n} \mathrm{Rep}_Y(\mathfrak{S}_n) \in \mathrm{Irrep}(GL(E))$

multiplicity $= \dim(\mathrm{Rep}_Y(\mathfrak{S}_n))$

$= \{\text{multiplicity of } \mathrm{Rep}_Y(\mathfrak{S}_n) \text{ in } \mathbb{K}(\mathfrak{S}_n)\}$

<u>Remark for latter purpose</u> :

$$T(E)/[E,[E,E]_\otimes]_\otimes$$

each Y occurs with multiplicity one

$\Rightarrow$ *model* for polynomial representations of $GL(E)$.

$\boxed{\text{Contractions in } E^Y}$

$T \in E^Y$ and $T' \in E^{*Y'}$
with $Y' \subset\subset Y \to \mathcal{C}(T|T') \in E^{\mathcal{C}(Y|Y')}$
given by tensor contraction of indices corresponding to the drawing of $\mathcal{C}(Y|Y')$

The fact that $\mathcal{C}(T|T')$ belong to $E^{\mathcal{C}(Y|Y')}$ is not completely obvious.

Multiforms

$\mathfrak{A} = \left(\otimes_{\text{gr}}^{N-1} \wedge \mathbb{R}^{D*}\right) \otimes C^{\infty}(\mathbb{R}^D)$={multiforms}
$C^{\infty}(\mathbb{R}^D)$-algebra generated by
$d_i x^\mu$; $i \in \{1, \ldots, N-1\}$, $\mu \in \{1, \ldots, D\}$
with relations
$d_i x^\mu d_j x^\nu + d_j x^\nu d_i x^\mu = 0$
or $\mathbb{R}$-algebra generated by $d_i x^\mu$ as above and $C^{\infty}(\mathbb{R}^D)$ with relations
$f d_i x^\mu - d_i x^\mu f = 0, \;\; f \in C^{\infty}(\mathbb{R}^D)$
$\mathfrak{A} = \oplus_{m_i \in \mathbb{N}} \mathfrak{A}^{m_1, \ldots, m_{N-1}}$
is multigraded, so also graded
$\mathfrak{A} = \oplus_n \mathfrak{A}^n$, $\mathfrak{A}^n = \oplus_{\sum m_i = n} \mathfrak{A}^{m_1, \ldots, m_{N-1}}$
and in fact graded-commutative.
There are $N-1$ antiderivations of graded $\mathbb{R}$-algebra d_i such that
$d_i f = d_i x^\mu \partial_\mu f \;\; (f \in C^{\infty}(\mathbb{R}^D)), d_i d_j x^\mu = 0$
$\Rightarrow d_i d_j + d_j d_i = 0$
$\Rightarrow d_I = \sum_{i \in I} d_i$ is a differential, i.e. an antiderivation of degree 1 with $d_I^2 = 0$
$\forall I \subset \{1, \ldots, N-1\}$ with $\# I \geq 1$.

$\boxed{\text{Generalized Poincaré lemma for } \mathfrak{A}}$

For $N=2$, one has $\mathfrak{A} = \Omega(\mathbb{R}^D) = \Omega_2(\mathbb{R}^D)$

THEOREM 2 *Let K be a non empty subset of $\{1,\dots,N-1\}$ and m be an integer $m \leq \#K$. Then*

$$\left(\prod_{i\in I} d_i\right)\omega = 0, \quad \forall I \subset K \text{ with } \#I = m$$

implies

$$\omega = \sum_{\substack{J\subset K \\ \#J=\#K-m+1}} \left(\prod_{j\in J} d_j\right)\alpha_J + \omega_0$$

with ω_0 polynomial of degree $\leq m-1$

For $N=2$ this is the Poincaré lemma and ω_0 can be incorporated in the differential.
This theorem which is interesting in itself is the main step for the proof of the generalized Poincaré lemma for $\Omega_N(\mathbb{R}^D)$.

Canonical inclusion $\Omega_N(\mathbb{R}^D) \subset \mathfrak{A}$

$E^Y \subset \wedge^{\tilde{m}_1} E \otimes \cdots \otimes \wedge^{\tilde{m}_c} E$
In particular
$E^{Y^N_{(N-1)n+i}} \subset (\otimes^i \wedge^{n+1} E) \otimes (\otimes^{N-1-i} \wedge^n E)$
decomposing the right-hand side into irreducible $GL(E)$ factors there is only one factor isomorphic to the left-hand side $\Rightarrow$ Image of $GL(E)$-invariant projection
$\Omega_N^{(N-1)n+i}(\mathbb{R}^D) = (\mathbb{R}^{D*})^{Y^N_{(N-1)n+i}} \otimes C^\infty(\mathbb{R}^D)$
$\Rightarrow \Omega_N(\mathbb{R}^D) = \mathrm{Im}(\pi)$
with π a $C^\infty(\mathbb{R}^D)$-linear GL_D-invariant homogeneous projection of $\mathfrak{A}$ onto itself.

LEMMA 4 *Let* $\omega \in \Omega_N^p(\mathbb{R}^D)$ *with* $p = (N-1)n + i$ *(*$0 \leq i < N-1$*). One has*

$$d\omega = c_p \pi(d_{i+1}\omega)$$

where $c_p \in \mathbb{R}\backslash\{0\}$*.*

$\boxed{\text{Theorem 2} \Rightarrow \text{Theorem 1}}$

LEMMA 5 *Let* $\omega \in \Omega_N^{(N-1)n}(\mathbb{R}^D)$. *One has*

$$d^k\omega = 0 \Leftrightarrow d_{i_1} \dots d_{i_k}\omega = 0$$

for $1 \leq k \leq N-1$, $\{i_1, \dots, i_k\} \subset \{1, \dots, N-1\}$.

In view of the symmetry in the columns, $d_{i_1} \dots d_{i_k}\omega = 0 \Leftrightarrow d_1 \dots d_k\omega = 0$. On the other hand $d_1 \dots d_k\omega \in \Omega_N(\mathbb{R}^N)$ because one has no component with first column of length $> n+1$.

LEMMA 6 *Let* $\omega \in \Omega_N^{(N-1)n}(\mathbb{R}^D)$ *with* $n \geq 1$. *Assume either that* ω *is polynomial of degree* $\leq k-1$ *or that one has*

$$\omega = \sum_{\substack{J \\ \#J=N-k}} (\prod_{j\in J} d_j)\alpha_J$$

Then $\omega = d^{N-k}\alpha$ *for some* $\alpha \in \Omega_N^{(N-1)n-N+k}(\mathbb{R}^D)$.

With these lemmas one deduces easily Theorem from Theorem 2.

$\boxed{\text{Appendix}}$

Y= Young diagram with $\tilde{m}_1 \geq \cdots \geq \tilde{m}_c$
Y'=Young diagram with $\tilde{m}'_1 \geq \cdots \geq \tilde{m}'_{c'}$
Set $Y > Y'$ whenever $\tilde{m}_p > \tilde{m}'_p$ and $\tilde{m}_k = \tilde{m}'_k$ for $k < p$ for some $p \geq 1$, with the convention $m_n = 0$ for $n > c$ ($\tilde{m}'_n = 0$ for $n > c'$).

PROPOSITION 1 *The relation $Y > Y'$ defines a total order on the set of Young diagrams. For (Y^N) one has*

$$Y_p^N = Inf\{Y | c < N \text{ and } |Y| = p\}$$

where $c = \#$ columns (Y) is the biggest integer with $\tilde{m}_c \neq 0$; one has of course $Y_{p+1}^N > Y_p^N$.

$\boxed{\text{VII - SIMPLICIAL } N\text{-COMPLEXES}}$

Ref : [22], [20], [19], [21].

Ref : [51], [52], [40], [43], [61].

$\boxed{\text{Presimplicial}}$

Presimplicial module : $(E_n)_{n\in\mathbb{N}}$
$\mathfrak{f}_i : E_n \to E_{n-1}$ for $i \in \{0, 1, \ldots, n\}$ (*faces*) such that
$\mathfrak{f}_i\mathfrak{f}_j = \mathfrak{f}_{j-1}\mathfrak{f}_i$ for $i > j$
($\sim$ relations of faces of simplices)
$d = \sum(-1)^i\mathfrak{f}_i : E_n \to E_{n-1}$
$(E = \oplus_n E_n, d)$ complex $\Rightarrow H(E) = \oplus_n H_n(E)$

Precosimplicial module : $(E^n)_{n\in\mathbb{N}}$
$\mathfrak{f}^i : E^n \to E^{n+1}$ for $i \in \{0, \ldots, n+1\}$ (*cofaces*)
$\mathfrak{f}^j\mathfrak{f}^i = \mathfrak{f}^i\mathfrak{f}^{j-1}$ for $i < j$ (duals)
$d = \sum(-1)^i\mathfrak{f}^i : E^n \to E^{n+1}$
$\Rightarrow$ cochain complex $\Rightarrow$ cohomology

$\mathrm{Hom}_{\mathbb{K}}(\bullet, \mathbb{K})$: Presimpl. $\to$ Precosimpl.

$\boxed{\text{Simplicial}}$

Simplicial module = Presimplicial module $(E_n)_{n\in\mathbb{N}}$, $\mathfrak{f}_i$ and *degeneracies* $\mathfrak{s}_i : E_n \to E_{n+1}$ for $i \in \{0, \ldots, n\}$ with

$$\mathfrak{s}_i\mathfrak{s}_j = \mathfrak{s}_{j+1}\mathfrak{s}_i \text{ for } i \leq j$$

$$\mathfrak{f}_i\mathfrak{s}_j = \begin{cases} \mathfrak{s}_{j-1}\mathfrak{f}_i \text{ for } i < j \\ \text{Identity for } i = j \ and \ i = j+1 \\ \mathfrak{s}_j\mathfrak{f}_{i-1} \text{ for } i > j+1 \end{cases}$$

Dually for *cosimplicial module* :
$\mathfrak{s}^i : E^n \to E^{n-1}$ $i \in \{0, \ldots, n-1\}$
$\mathfrak{s}^j\mathfrak{s}^i = \mathfrak{s}^i\mathfrak{s}^{j+1}$ for $i \leq j$

$$\mathfrak{s}^j\mathfrak{f}^i = \begin{cases} \mathfrak{f}^i\mathfrak{s}^{j-1} \text{ for } i < j \\ \text{Identity for } i = j \text{ and } i = j+1 \\ \mathfrak{f}^{i-1}\mathfrak{s}^j \text{ for } i > j+1 \end{cases}$$

$\boxed{\text{Associated } N\text{-complexes}}$

$$\begin{cases} (E^n)_{n\in\mathbb{N}} = \text{ Precosimplicial module} \\ \mathbb{K} \text{ and } q \in \mathbb{K} \text{ satisfy } (A_0), \text{ i.e. } [N]_q = 0 \end{cases}$$

$$\begin{aligned} d_0 &= \textstyle\sum_{k=0}^{n+1} q^k \mathfrak{f}^k : E^n \to E^{n+1} \\ d_1 &= \textstyle\sum_{k=0}^{n} q^k \mathfrak{f}^k - q^n \mathfrak{f}^{n+1} \end{aligned}$$

$d_m =$
$\sum_{k=0}^{n-m+1} q^k \mathfrak{f}^k - q^{n-m+1} \left(\sum_{p=0}^{m-1} (-1)^p \mathfrak{f}^{n-m+2+p} \right)$
for $n \geq m-1$
$d_m = d$ for $n \leq m-1$

LEMMA 1 *One has* $d_m^N = 0, \ \forall m \in \mathbb{N}$

Cumbersome induction (easy for the parts $\delta_m = \sum_{k=0}^{n-m+1} q^k \mathfrak{f}^k, \ n \geq m-1$).

Notice that $d_1 = d : E^0 \to E^1$.
Similar N-differentials in the simplicial case.

$\boxed{\text{Expressing } H_{(k)}(d_m) \text{ in terms of } H(d)}$

$$H_{(k)}(d_m) = H_{(k)}(E, d_m), \;\; H(d) = H(E, d)$$

$\mathbb{K}$ and $q \in \mathbb{K}$ satisfy (A_1), E cosimplicial

THEOREM 1

$$(0) \qquad \begin{cases} H^{Nr-1}_{(k)}(d_0) = H^{2r-1}(d) \\ H^{N(r+1)-k-1}_{(k)}(d_0) = H^{2r}(d) \\ H^{n}_{(k)}(d_0) = 0 \;\; \textit{otherwise} \end{cases}$$

$$(1) \qquad \begin{cases} H^{Nr}_{(k)}(d_1) = H^{2r}(d) \\ H^{N(r+1)-k}_{(k)}(d_1) = H^{2r+1}(d) \\ H^{n}_{(k)}(d_1) = 0 \;\; \textit{otherwise} \end{cases}$$

More generally under the same assumption

THEOREM 2 *Setting* $E_m = \oplus_{n \geq m-1} E^n$

$$H^{Nr+m-1}_{(k)}(E_m, d_m) = H^{2r+m-1}(d)$$

for $r \geq 1$,

$$H^{N(r+1)-k+m-1}_{(k)}(E_m, d_m) = H^{2r+m}(d)$$

$$H^{m-1}_{(k)}(E_m, d_m) = \mathit{Ker}(d : E^{m-1} \to E^m)$$

and $H^n_{(k)}(E_m, d_m) = 0$ *otherwise.*

In particular for $n \geq N - k + m - 1$ one has $H^n_{(k)}(d_m) = 0$ if $n \neq m - 1 \mod (N)$ or $n + k \neq m - 1 \mod (N)$ and

$$H^{Nr+m-1}_{(k)}(d_m) = H^{2r+m-1}(d) \quad \forall r \geq 1,$$

$$H^{N(r+1)-k+m-1}_{(k)}(d_m) = H^{2r+m}(d).$$

$\boxed{\text{Simplicial case}}$

In the case where E is simplicial one defines similarily the sequence of N-differentials d_m in terms of faces and, under (A_1) one has

THEOREM 3

$$(0)\begin{cases} H_{(k),Nr-1}(d_0) = H_{2r-1}(d) \\ \\ H_{(k),Nr+k-1}(d_0) = H_{2r}(d) \\ \\ H_{(k),n}(d_0) = 0 \ \textit{otherwise} \end{cases}$$

$$(1)\begin{cases} H_{(k),Nr}(d_1) = H_{2r}(d) \\ \\ H_{(k),Nr+k}(d_1) = H_{2r+1}(d) \\ \\ H_{(k),n}(d_1) = 0 \ \textit{otherwise} \end{cases}$$

$\boxed{\text{Products}}$

$\mathfrak{M}$-*precosimplicial* module = precosimplicial module $(E^n, \mathfrak{f}^i)$ with $\mathbb{K}$-linear associative product $E^a \otimes E^b \to E^{a+b}, \;\; \alpha \otimes \beta \mapsto \alpha\beta$ such that

$$\mathfrak{f}^i(\alpha\beta) = \begin{cases} \mathfrak{f}^i(\alpha)\beta & \text{if } \; i \leq a \\ \alpha\mathfrak{f}^{i-a}(\beta) & \text{if } \; i > a \end{cases}$$

for $i \in \{0, \ldots, a+b+1\}$ and $\mathfrak{f}^{a+1}(\alpha)\beta = \alpha\mathfrak{f}^0(\beta)$ for $\alpha \in E^a, \; \beta \in E^b$.

PROPOSITION 1 *Let (E^n) be $\mathfrak{M}$- precosimplicial with $\mathbb{K}$ and $q \in \mathbb{K}$ satisfying (A_0). Then $(E = \oplus_n E^n, d_1)$ is a graded q-differential algebra.*

In particular for $N = 2$ $(q = -1)$ it is a graded differential algebra.

$\boxed{\text{Examples}}$

$(C^n(\mathcal{A},\mathcal{A}))$ with usual product $(\otimes_{\mathcal{A}})$

$\mathfrak{f}^0(\omega)(x_0,\dots,x_n) = x_0\omega(x_1,\dots,x_n)$

$\mathfrak{f}^i(\omega)(x_0,\dots,x_n) = \omega(x_0,\dots,x_{i-1}x_i,\dots,x_n)$,
$1 \le i \le n$

$\mathfrak{f}^{n+1}(\omega)(x_0,\dots,x_n) = \omega(x_0,\dots,x_{n-1})x_n$

$(\mathfrak{T}^n(\mathcal{A}))$ with its product

$\mathfrak{f}^0(x_0\otimes\cdots\otimes x_n) = \mathbb{1}\otimes x_0\otimes\cdots\otimes x_n$

$\mathfrak{f}^i(x_0\otimes\cdots\otimes x_n) =$
$x_0\otimes\cdots\otimes x_{i-1}\otimes\mathbb{1}\otimes x_i\otimes\cdots\otimes x_n, \quad 1\le i\le n$

$\mathfrak{f}^{n+1}(x_0\otimes\cdots\otimes x_n) = x_0\otimes\cdots\otimes x_n\otimes\mathbb{1}$

One has $\Psi\circ\mathfrak{f}^i = \mathfrak{f}^i\circ\Psi : \mathfrak{T}^n(\mathcal{A}) \to C^n(\mathcal{A},\mathcal{A})$

$\boxed{\mathfrak{M}\text{-cosimplicial}}$

A $\mathfrak{M}$-precosimplicial module (E^n) is $\mathfrak{M}$-*cosimplicial* if it is a cosimplicial with codegeneracies satisfying

$$\mathfrak{s}^i(\alpha\beta) = \begin{cases} \mathfrak{s}^i(\alpha)\beta & \text{if} \quad i < a \\ \alpha\mathfrak{s}^{i-a}(\beta) & if \quad i \geq a \end{cases}$$

for $i \in \{0, \ldots, a+b-1\}, \ \alpha \in E^a, \ \beta \in E^b$.

$(C^n(\mathcal{A}, \mathcal{A}))$ is $\mathfrak{M}$-cosimplicial with

$$\mathfrak{s}^i(\omega)(x_1, \ldots, x_{n-1}) = \omega(x_1, \ldots, x_i, \mathbb{1}, x_{i+1}, \ldots, x_{n-1})$$

$(\mathfrak{T}^n(\mathcal{A}))$ is $\mathfrak{M}$-cosimplicial with

$$\mathfrak{s}^i(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$

PROPOSITION 2 Ψ *is a* $\mathfrak{M}$*-cosimplicial homomorphism.*

i.e. one also has $\Psi \circ \mathfrak{s}^i = \mathfrak{s}^i \circ \Psi$

$\boxed{\text{Normalization}}$

- $(E^n), \mathfrak{f}^i, \mathfrak{s}^j$ cosimplicial module

$\rightarrow (E, d)$ corresponding complex $N(E^n) = \cap \mathrm{Ker}(\mathfrak{s}^j)$ stable by d

$\rightarrow (N(E), d)$ and $H(N(E)) = H(E)$

- In the case (E^n) $\mathfrak{M}$-cosimplicial one has :

PROPOSITION 3 *Let (E^n) be a $\mathfrak{M}$-cosimplicial module. Then (E, d) is a graded differential algebra and $(N(E), d)$ is a graded differential subalgebra of (E, d).*

The first part is a particular case of Proposition 1 ($q = -1, N = 2$).

Notice that $N(\mathfrak{T}(\mathcal{A})) = \Omega(\mathcal{A})$ is the universal differential envelope of $\mathcal{A}$ and that Ψ is a homomorphism of $\Omega(\mathcal{A})$ into $(NC(\mathcal{A}, \mathcal{A}), d)$

VIII - A N-DIFFERENTIAL B.R.S. PROBLEM

Ref : [22], [27], [28].

A N-differential BRS-like problem

For the zero modes of the $SU(2)$ WZNW model, $N=$ height of the current algebra representation $= k+2$ with $k=$ Kac-Moody level.

- $(\mathcal{H}, A) = N$-differential space ($A^N = 0$)

- $\mathcal{U}_q$ acts on $\mathcal{H}$ $\quad (q^N = -1)$, $[\mathcal{U}_q, A] = 0$

- $\mathcal{H}_I = \{\mathcal{U}_q\text{-invariant} \in \mathcal{H}\} \Rightarrow (\mathcal{H}_I, A) = N$-diff.

- $\mathcal{H}_{phys} \simeq \oplus_{k=1}^{N-1} H_{(k)}(\mathcal{H}_I, A)$

Problem : Avoid the restriction to $\mathcal{H}_I$ i.e. find a N-differential space such that $\mathcal{U}_q$-invariance is captured in the N-differential, etc.

$\boxed{N\text{-complex of linear inclusion}}$

$E_1 \subset E$ vector spaces $\Rightarrow (E_0, \delta_0) N$-complex

$E_0 = \oplus_{n=0}^{N-1} E_0^n$, $E_0^0 = E$ and $E_0^n = E/E_1$ $n \geq 1$

$\delta_0 : E_0^n \to E_0^{n+1}$, $\delta_0 = \pi : E_0^0 = E \to E/E_1 = E_0^1$

$\delta_0 = \mathrm{Id} : E_0^n = E/E_1 \to E/E_1 = E_0^{n+1}$

for $1 \leq n \leq N-1$ and $\delta_0 = 0$ on E_0^{N-1}.

PROPOSITION 1 *One has $H^n_{(k)}(E_0, \delta_0) = 0$ for $n \geq 1$ and $H^0_{(k)}(E_0, \delta_0) = E_1$ for any $k \in \{1, \dots, N-1\}$.*

(E_0, δ_0) is characterized by the following .

PROPOSITION 2 *Any linear $\alpha : E \to C^0$ where $(C = \oplus_{n\in\mathbb{N}} C^n, d)$ is a cochain N-complex such that $d \circ \alpha(E_1) = 0$ has a unique extension as a homomorphism $\bar{\alpha} : (E_0, \delta_0) \to (C, d)$ of N-complexes.*

$\boxed{\text{Inclusion of } N\text{-differential spaces}}$

(E, δ_1) N-differential $E_1 \subset E$ with $\delta_1 E_1 \subset E_1$
$q =$ primitive N^{th} root of unit.

PROPOSITION 3 $\delta_1 : E_0^0 = E \to E = E_0^0$ *has a unique linear extension, denoted again by* $\delta_1 : E_0 \to E_0$*, which is homogeneous degree 0 and satisfies* $\delta_1\delta_0 = q\delta_0\delta_1$*. One has* $\delta_1^N = 0$ *and* $(\delta_0 + \delta_1)^N = 0$ *on* E_0*.*

Use Proposition 2. Take $\alpha = \delta_1 : E \to E_0^0 \Rightarrow \bar{\delta}_1$ and set $\delta_1 = q^{\text{degree}}\bar{\delta}_1$.

THEOREM 1 $H_{(k)}(E_1, \delta_1) = H_{(k)}(E_0, \delta_0 + \delta_1)$ *for* $k \in \{1, \dots, N-1\}$

One has the short exact sequence

$$0 \to (E_1, \delta_1) \xrightarrow{\subseteq} (E_0, \delta_0 + \delta_1) \to (F, \delta) \to 0$$

and $H_{(k)}(F, \delta) = 0$ for $k \in \{0, 1, \dots, N-1\} \Rightarrow$

$$0 \xrightarrow{\partial} H_{(k)}(E_1, \delta_1) \xrightarrow{\simeq} H_{(k)}(E_0, \delta_0 + \delta_1) \to 0$$

Hopf algebra action

Assume now that a Hopf algebra $\mathcal{U}$ acts on (E, δ_1) by automorphism and that E_1 is the set of $\mathcal{U}$-invariant elements of E.

LEMMA 1 *For $x \in E = C^0(\mathcal{U}, E)$ the following statements (i), (ii) and (iii) are equivalent*
(i) $d_1^k(x) = 0$ *for some* $k \in \{1, \ldots, N-1\}$
(ii) $x \in E_1$
(iii) $d_1^n(x) = 0$ *for any* $n \in \{1, \ldots, N-1\}$

On $C^0(\mathcal{U}, E)$ one has $d_1 = d$ and by induction $d_1^n(x)(\mathbb{1}, \ldots, \mathbb{1}, X) = [n]_q X(x) \Rightarrow$ Lemma 1.

PROPOSITION 4 (E_0, δ_0) *identifies with the N-subcomplex of $(C(\mathcal{U}, E), d_1)$ generated by E.*

i.e. $E_0 = E \oplus d_1 E \oplus \cdots \oplus d_1^{N-1} E$, $d_1 \restriction E_0 = \delta_0$.

The homomorphism $(E_0, \delta_0) \to (C(\mathcal{U}, E), d_1)$ of Proposition 2 is injective by Lemma 1.

$$\boxed{H_{(k)}(E_1,\delta_1) \text{ in terms of } C(\mathcal{U},E)}$$

LEMMA 2 *One extends* δ_1 *from* E_0 *to* $C(\mathcal{U},E)$ *by* $(\delta_1\omega)(X_1,\dots,X_n) = q^n\delta_1\omega(X_1,\dots,X_n)$, $\omega \in C^n(\mathcal{U},E)$ *and one has* $\delta_1 d_1 = q d_1 \delta_1$, $\delta_1^N = (d_1+\delta_1)^N = 0$.

Let the filtration F^n of $H_{(k)}(C(\mathcal{U},E),d_1+\delta_1)$ be defined by $F^nH_{(k)}(C(\mathcal{U},E),d_1+\delta_1) = [\mathrm{Ker}(d_1+\delta_1)^k \cap \oplus_{0\leq r\leq n} C^r(\mathcal{U},E)]$ for $n \geq 0$ and by 0 for $n<0$.

THEOREM 2 *The inclusion* $E_0 \subset C(\mathcal{U},E)$ *induces isomorphisms*

$$H_{(k)}(E_0,\delta_0+\delta_1) \simeq F^0H_{(k)}(C(\mathcal{U},E),d_1+\delta_1)$$

for $k \in \{1,\dots,N-1\}$.

In particular, one has for $1 \leq k \leq N-1$

$$F^0H_{(k)}(C(\mathcal{U},E),d_1+\delta_1) \simeq H_{(k)}(E_1,\delta_1)$$

IX - HOMOGENEOUS ALGEBRAS

Ref : [8], [26].

Ref : [6], [7], [53], [47], [48], [49].

$\boxed{\text{Homogeneous Algebras}}$

$\mathbb{K}$ field of characteristic zero
$N \in \mathbb{N}$ with $N \geq 2$

$\mathcal{A}$ *N-homogeneous algebra* :
$\mathcal{A} = A(E, R) = T(E)/(R)$
$\dim(E) < \infty$, $R \subset E^{\otimes^N}$

$\Rightarrow \mathcal{A}$ connected graded algebra ($\mathcal{A}_0 = \mathbb{K}\mathbb{1}$)
generated in degree 1 ($\mathcal{A}_1 = E$).

$f : A(E, R) \to A(E', R')$ *morphism* :
$f \in \mathrm{Hom}_{\mathbb{K}}(E, E')$ such that $f^{\otimes^N}(R) \subset R'$
$\Rightarrow f$ induces an algebra homomorphism.

Category $\mathbf{H_N Alg}$
Forgetful functor $\mathcal{A} \mapsto E$ from $\mathbf{H_N Alg}$ to $\mathbf{Vect}$

$\boxed{\text{Duality}}$

$\mathcal{A} = A(E, R)$ N-homogeneous algebra
$\mapsto \mathcal{A}^! = A(E^*, R^\perp)$ *dual N-homogeneous algebra*

where

$R^\perp = \{\omega \in (E^{\otimes^N})^* | \omega(x) = 0, \ \forall x \in R\}$
with the identification $(E^{\otimes^N})^* = E^{*\otimes^N}$

$(\mathcal{A}^!)^! = \mathcal{A}$
$f : \mathcal{A} \to \mathcal{A}'$ morphism
$\mapsto f^! : (\mathcal{A}')^! \to \mathcal{A}^!$ morphism

$(\mathcal{A} \mapsto \mathcal{A}^!, f \mapsto f^!)$ involutive contravariant functor
$\mathcal{A} \mapsto \mathcal{A}^!$ is a lifting to $\mathbf{H_N Alg}$ of the duality $E \mapsto E^*$ in $\mathbf{Vect}$

$\boxed{\text{Products}}$

$\mathcal{A} = A(E, R)$, $\mathcal{A}' = A(E', R')$

$\mathcal{A} \circ \mathcal{A}' = A(E \otimes E', \pi_N(R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R'))$
$\mathcal{A} \bullet \mathcal{A}' = A(E \otimes E', \pi_N(R \otimes R'))$
$\pi_N : (1, 2, \ldots, 2N) \mapsto (1, N+1, \ldots, N, 2N)$
acting on the factors of $\otimes$.

$(R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R')^\perp = R^\perp \otimes R'^\perp \Rightarrow$
$(\mathcal{A} \circ \mathcal{A}')^! = \mathcal{A}^! \bullet \mathcal{A}'^!$
$(\mathcal{A} \bullet \mathcal{A}')^! = \mathcal{A}^! \circ \mathcal{A}'^!$

$R \otimes R' \subset R \otimes E'^{\otimes^N} + E^{\otimes^N} \otimes R' \Rightarrow$
$p : \mathcal{A} \bullet \mathcal{A}' \to \mathcal{A} \circ \mathcal{A}'$
epimorphism of $\mathbf{H_N Alg}$ ($p = I_{E \otimes E'}$).

$\circ$ et $\bullet$ are lifting
to $\mathbf{H_N Alg}$ of $\otimes$ in $\mathbf{Vect}$.

$\boxed{\text{Connections with } \mathcal{A}\otimes\mathcal{A}'}$

If $N \geq 3$, $\mathcal{A}\otimes\mathcal{A}'$ is not N-homogeneous.

$\tilde{\imath} : T(E\otimes E') \to T(E)\otimes T(E')$
$\tilde{\imath} = \pi_n^{-1} : (E\otimes E')^{\otimes^n} \to E^{\otimes^n}\otimes E'^{\otimes^n}$

$\tilde{\imath}$ is an injective algebra-homomorphism
$\tilde{\imath}(T(E\otimes E')) = \oplus_n E^{\otimes^n}\otimes E'^{\otimes^n}$

PROPOSITION 1 $\mathcal{A} = A(E,R)$, $\mathcal{A}' = A(E',R')$
$\tilde{\imath}$ *induces an injective algebra-homomorphism*

$$i : \mathcal{A}\circ\mathcal{A}' \to \mathcal{A}\otimes\mathcal{A}'$$

and

$$i(\mathcal{A}\circ\mathcal{A}') = \oplus_n \mathcal{A}_n\otimes\mathcal{A}'_n$$

$\boxed{\text{Units}}$

$$\mathcal{A} = A(E,R),\ \mathcal{A}' = A(E',R'),\ \mathcal{A}'' = A(E'',R'')$$

$$(E\otimes E')\otimes E'' \simeq E\otimes(E'\otimes E'')$$

induces $(\mathcal{A}\circ\mathcal{A}')\circ\mathcal{A}'' \simeq \mathcal{A}\circ(\mathcal{A}'\circ\mathcal{A}'')$
$E\otimes E' \simeq E'\otimes E$ induces $\mathcal{A}\circ\mathcal{A}' \simeq \mathcal{A}'\circ\mathcal{A}$
$\mathbb{K} = \mathbb{K}t$ unit object of $(\mathbf{Vect},\otimes)$
$\mapsto \mathbb{K}[t] = A(\mathbb{K}t, 0) \simeq T(\mathbb{K})$ unit
object of $(\mathbf{H_N Alg},\circ)$.

THEOREM 1 $(\imath)$ $\mathbf{H_N Alg}$ *endowed with* $\circ$ *is a tensor category with unit object* $\mathbb{K}[t]$
$(\imath\imath)$ $\mathbf{H_N Alg}$ *endowed with* $\bullet$ *is a tensor category unit object* $\wedge_N\{d\} = \mathbb{K}[t]^!$

$(\imath) \Leftrightarrow (\imath\imath)$ by duality
$\wedge_N\{d\}$ = unital graded algebra generated in degree one by d with relation $d^N = 0$.

$$\boxed{\textbf{Hom}(\mathcal{A}, \mathcal{B})}$$

THEOREM 2

$$\mathit{Hom}_{\mathbb{K}}(E \otimes E', E'') = \mathit{Hom}_{\mathbb{K}}(E, E'^{*} \otimes E'')$$

in $\mathbf{Vect}$ *induces*

$$\mathit{Hom}(\mathcal{A} \bullet \mathcal{A}', \mathcal{A}'') = \mathit{Hom}(\mathcal{A}, \mathcal{A}'^{!} \circ \mathcal{A}'')$$

in $\mathbf{H_N Alg}$.

$\Rightarrow$ internal **Hom** for $(\mathbf{H_N Alg}, \bullet)$
$\textbf{Hom}(\mathcal{A}', \mathcal{A}'') = \mathcal{A}'^{!} \circ \mathcal{A}''$

The canonical mappings $(E^{*} \otimes E') \otimes E \to E'$ and $(E'^{*} \otimes E'') \otimes (E^{*} \otimes E') \to E^{*} \otimes E''$ induce products $\mu : \textbf{Hom}(\mathcal{A}, \mathcal{A}') \bullet \mathcal{A} \to \mathcal{A}'$
$m : \textbf{Hom}(\mathcal{A}', \mathcal{A}'') \bullet \textbf{Hom}(\mathcal{A}, \mathcal{A}') \to \textbf{Hom}(\mathcal{A}, \mathcal{A}'')$
with obvious associativity properties.

$$\boxed{\mathbf{end}(\mathcal{A})}$$

Setting $\mathbf{hom}(\mathcal{A},\mathcal{B}) = \mathbf{Hom}(\mathcal{A}^!,\mathcal{B}^!)^! = \mathcal{A}^! \bullet \mathcal{B}$ by duality from μ, m one obtains

$$\delta_0 : \mathcal{B} \to \mathbf{hom}(\mathcal{A},\mathcal{B}) \circ \mathcal{A}$$

$$\Delta_0 : \mathbf{hom}(\mathcal{A},\mathcal{C}) \to \mathbf{hom}(\mathcal{B},\mathcal{C}) \circ \mathbf{hom}(\mathcal{A},\mathcal{B})$$

with induce via i

$$\delta : \mathcal{B} \to \mathbf{hom}(\mathcal{A},\mathcal{B}) \otimes \mathcal{A}$$

$$\Delta : \mathbf{hom}(\mathcal{A},\mathcal{C}) \to \mathbf{hom}(\mathcal{B},\mathcal{C}) \otimes \mathbf{hom}(\mathcal{A},\mathcal{B})$$

with obvious coassociativity properties.

THEOREM 3

$$\mathbf{end}(\mathcal{A}) = \mathcal{A}^! \bullet \mathcal{A}$$

endowed with Δ *is a bialgebra with counit* $\varepsilon : \mathcal{A}^! \bullet \mathcal{A} \to \mathbb{K}$ *induced by the duality* $\varepsilon = \langle \cdot, \cdot \rangle : E^* \otimes E \to \mathbb{K}$ *and* $\mathcal{A}$ *endowed with* δ *is an* $\mathbf{end}(\mathcal{A})$*-comodule.*

$\boxed{N\text{-complex } L(f)}$

Theorem 1 (ii) + Theorem 2 $\Rightarrow$
$\mathrm{Hom}(\mathcal{B},\mathcal{C}) = \mathrm{Hom}(\wedge_N\{d\}, \mathcal{B}^! \circ \mathcal{C})$

$\xi_f \in \mathcal{B}^! \circ \mathcal{C}$ image of d corresponding to $f \in \mathrm{Hom}(\mathcal{B},\mathcal{C})$. One has $(\xi_f)^N = 0$.

$d=$ Left multiplication by $i(\xi_f)$ in $\mathcal{B}^! \otimes \mathcal{C}$. $d^N = 0 \Rightarrow$

$L(f) = (\mathcal{B}^! \otimes \mathcal{C}, d)$ is a cochain N-complex of right $\mathcal{C}$-modules :

$$d : \mathcal{B}^!_n \otimes \mathcal{C} \to \mathcal{B}^!_{n+1} \otimes \mathcal{C}$$

When $\mathcal{A} = \mathcal{B} = \mathcal{C}$, $f = I_{\mathcal{A}}$ one denotes it by $L(\mathcal{A})$

N-complex $K(f)$ - I

Apply $\mathrm{Hom}_{\mathcal{C}}(.\,,\mathcal{C})$ to each right $\mathcal{C}$-module of $L(f) = (\mathcal{B}^! \otimes \mathcal{C}, d) \Rightarrow$
The chain N-complex $K(f)$ of left $\mathcal{C}$-modules.

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{B}^!_n \otimes \mathcal{C}, \mathcal{C}) \simeq \mathcal{C} \otimes (\mathcal{B}^!_n)^*$$

$\Rightarrow K(f) = (\mathcal{C} \otimes \mathcal{B}^{!*}, d)$,
$d : \mathcal{C} \otimes (\mathcal{B}^!_{n+1})^* \to \mathcal{C} \otimes (\mathcal{B}^!_n)^*$

When $\mathcal{A} = \mathcal{B} = \mathcal{C}$, $f = I_{\mathcal{A}}$
one denotes it by $K(\mathcal{A})$

We shall describe an alternative construction for $K(f)$.

$\boxed{\text{Disgression}}$

LEMMA 1 *A associative algebra with product m, C coassociative coalgebra with coproduct Δ, Hom$_{\mathbb{K}}(C, A)$ endowed with the convolution product*

$$\alpha * \beta = m \circ (\alpha \otimes \beta) \circ \Delta, (\alpha, \beta \in \mathit{Hom}_{\mathbb{K}}(C, A))$$

For $\alpha \in \mathit{Hom}_{\mathbb{K}}(C, A)$ define

$$d_\alpha \in \mathit{End}_A(A \otimes C) = \mathit{Hom}_A(A \otimes C, A \otimes C)$$

as the composite

$$A \otimes C \xrightarrow{I_A \otimes \Delta} A \otimes C \otimes C \xrightarrow{I_A \otimes \alpha \otimes I_C} A \otimes A \otimes C \xrightarrow{m \otimes I_C} A \otimes C$$

Then $\alpha \mapsto d_\alpha$ is an algebra homomorphism i.e. $d_{\alpha * \beta} = d_\alpha \circ d_\beta$.

$\mathbb{1}$ unity of A, ε counit of C
$\Rightarrow \alpha \mapsto \varepsilon(\alpha)\mathbb{1}$ unit of Hom$_{\mathbb{K}}(C, A)$

structure of left A module

$$x(a \otimes c) = xa \otimes c$$

$\boxed{N\text{-complex } K(f) \text{ - II}}$

$$\mathcal{B} = A(E, R), \mathcal{C} = A(E', R')$$

$(\mathcal{B}^!)^*$ coalgebra with $(\mathcal{B}^!_1)^* = \mathcal{B}_1 = E$

$$f \in \mathrm{Hom}(\mathcal{B}, \mathcal{C}) \mapsto \alpha \in \mathrm{Hom}_{\mathbb{K}}((\mathcal{B}^!)^*, \mathcal{C})$$

$\alpha = f : E \to E'$ in degree 1 and $\alpha = 0$ otherwise
$\alpha^{*N} = \underbrace{\alpha * \cdots * \alpha}_{N}$ is the composite

$$R \xrightarrow{f^{\otimes N}} E'^{\otimes N} \longrightarrow E'^{\otimes N}/R'$$

$\Rightarrow \alpha^{*^N} = 0$ $(f(R) \subset R') \Rightarrow$
$(\mathcal{C} \otimes \mathcal{B}^{!*}, d_\alpha)$ is a chain N-complex of left $\mathcal{C}$-modules which coincides with $K(f)$ $(d = d_\alpha)$.

$\boxed{\text{Components}}$

$\mathcal{B}^!_n = E^{*\otimes^n}$ if $n < N$
$\mathcal{B}^!_n = E^{*\otimes^n} / \sum_{r+s=n-N} E^{*\otimes^r} \otimes R^\perp \otimes E^{*\otimes^s}$ if $n \geq N$
$\Rightarrow$
$(\mathcal{B}^!_n)^* = E^{\otimes^n}$ if $n < N$
$(\mathcal{B}^!_n)^* = \cap_{r+s=n-N} E^{\otimes^r} \otimes R \otimes E^{\otimes^s}$ if $n \geq N$
Thus in any case one has

$$(\mathcal{B}^!_n)^* \subset E^{\otimes^n}$$

The N-differential d of $K(f)$ is induced by

$c \otimes (e_1 \otimes e_2 \otimes \cdots \otimes e_n) \to cf(e_1) \otimes (e_2 \otimes \cdots \otimes e_n)$

of $\mathcal{C} \otimes E^{\otimes^n}$ into $\mathcal{C} \otimes E^{\otimes^{n-1}}$

$\boxed{\text{Splitting of } K(f)}$

$K(f)$ splits into sub-N-complexes.
$K(f)^n = \oplus_m \mathcal{C}_{n-m} \otimes (\mathcal{B}^!_m)^*$, $n \in \mathbb{N}$
homogeous for the total degree.
$K(f)^0$ is $0 \to \mathbb{K} \to 0$
$K(f)^n$ is

$$0 \to E^{\otimes^n} \xrightarrow{f\otimes I_E^{\otimes^{n-1}}} E' \otimes E^{\otimes^{n-1}} \to \cdots \xrightarrow{I_{E'}^{\otimes^{n-1}}\otimes f} E'^{\otimes^n} \to 0$$

for $1 \leq n \leq N-1$ and $K(f)^N$ is

$$0 \to R \to E' \otimes E^{\otimes^{N-1}} \to \dots$$

$$\cdots \to E'^{\otimes^{N-1}} \otimes E \stackrel{can}{\to} E'^{\otimes^N}/R' \to 0$$

These $K(f)^n$ for $1 \leq n \leq N-2$ are acyclic only if $E = E' = 0$.

LEMMA 2 *$K(f)^{N-1}$ and $K(f)^N$ are acyclic if and only if f is an isomorphism (of $\mathbf{H_N Alg}$).*

$\boxed{\text{Maximal acyclicity}}$

The maximal acyclicity for $K(f)$ that can be a priori expected is the acyclicity of the N-complexes $K(f)^n$ for $n \geq N-1$.

Lemme 2 $\Rightarrow$ One can restrict attention to $K(\mathcal{A})$

PROPOSITION 2 $N \geq 3$
$K(\mathcal{A})^n$ *acyclic* $\forall n \geq N-1 \Leftrightarrow R = 0$ *or* $R = E^{\otimes^N}$

Thus the assumption of Proposition 2 leads for $N \geq 3$ to a trivial class of algebras although for $N = 2$ this assumption characterizes the quadratic Koszul algebra. Generalization? i.e. nontrivial maximal acyclicity?

$\boxed{\text{Koszul Homogeneous Algebras}}$

$$C_{p,r}, \quad 0 \leq r \leq N-2, \; r+1 \leq p \leq N-1$$

complexes obtained by *contraction* of $K(\mathcal{A})$

$$\dots \overset{d^{N-p}}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{nN+r} \overset{d^p}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{nN-p+r} \overset{d^{N-p}}{\to} \dots$$

$$\dots \overset{d^p}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{N-p+r} \overset{d^{N-p}}{\to} \mathcal{A} \otimes \mathcal{A}^{!*}_{r} \to 0$$

PROPOSITION 3 $N \geq 3$, $(p,r) \neq (N-1,0)$

$$H_1(C_{p,r}) = 0 \Rightarrow R = 0 \textit{ or } R = E^{\otimes^N}.$$

$\mathcal{A}$ *Koszul N-homogeneous algebra* :

$$H_n(C_{N-1,0}) = 0, \quad \forall n \geq 1.$$

$\Rightarrow$ resolution of the trivial $\mathcal{A}$-module $\mathbb{K}$.
$C_{N-1,0}$ will be denoted by $\mathcal{K}(\mathcal{A},\mathbb{K})$, it coincides with the *Koszul complex* introduced by Roland Berger.

$\boxed{\text{Complex } \mathcal{K}(\mathcal{A}, \mathcal{A})}$

$K(\mathcal{A})$ N-complex of left $\mathcal{A}$-modules

$$\cdots \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \xrightarrow{d} \mathcal{A} \otimes \mathcal{A}^{!*}_{n} \xrightarrow{d} \cdots$$

d induced by

$$a \otimes (e_1 \otimes \cdots \otimes e_{n+1}) \mapsto ae_1 \otimes (e_2 \otimes \cdots \otimes e_{n+1})$$

$\tilde{K}(\mathcal{A})$ N-complex of right $\mathcal{A}$-modules

$$\cdots \xrightarrow{\tilde{d}} \mathcal{A}^{!*}_{n+1} \otimes \mathcal{A} \xrightarrow{\tilde{d}} \mathcal{A}^{!*}_{n} \otimes \mathcal{A} \xrightarrow{\tilde{d}} \cdots$$

$\tilde{d}$ induced by

$$(e_1 \otimes \cdots \otimes e_{n+1}) \otimes a \mapsto (e_1 \otimes \cdots \otimes e_n) \otimes e_{n+1}a$$

$\Rightarrow$ two N-complexes of bimodules (L, R)

$$\cdots \xrightarrow{d_L, d_R} \mathcal{A} \otimes \mathcal{A}^{!*}_{n+1} \otimes \mathcal{A} \xrightarrow{d_L, d_R} \mathcal{A} \otimes \mathcal{A}^{!*}_{n} \otimes \mathcal{A} \xrightarrow{d_L, d_R} \cdots$$

$$d_L = d \otimes I_{\mathcal{A}}, \qquad d_R = I_{\mathcal{A}} \otimes \tilde{d}$$

...

$\boxed{\text{Complex } \mathcal{K}(\mathcal{A},\mathcal{A})}$, *continuation*

$$d_L d_R = d_R d_L \Rightarrow (d_L - d_R)\left(\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1}\right) =$$

$$\left(\sum_{p=0}^{N-1} d_L^p d_R^{N-p-1}\right)(d_L - d_R) = d_L^N - d_R^N = 0$$

Define the chain complex of $(\mathcal{A},\mathcal{A})$-bimodules $\mathcal{K}(\mathcal{A},\mathcal{A})$ by

$$\mathcal{K}_{2m}(\mathcal{A},\mathcal{A}) = \mathcal{A}\otimes\mathcal{A}^{!*}_{Nm}\otimes\mathcal{A} = \mathcal{K}_{2m}(\mathcal{A},\mathbb{K})\otimes\mathcal{A}$$

$$\mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A}) = \mathcal{A}\otimes\mathcal{A}^{!*}_{Nm+1}\otimes\mathcal{A} = \mathcal{K}_{2m+1}(\mathcal{A},\mathbb{K})\otimes\mathcal{A}$$

with differential δ' defined by

$$\delta' = d_L - d_R : \mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A}) \to \mathcal{K}_{2m}(\mathcal{A},\mathcal{A})$$

$$\delta' = \sum_{p=0}^{N-1} d_L^p d_R^{N-p-1} : \mathcal{K}_{2(m+1)}(\mathcal{A},\mathcal{A}) \to \mathcal{K}_{2m+1}(\mathcal{A},\mathcal{A})$$

Properties of Koszul algebras

PROPOSITION 4 $\mathcal{A} = A(E,R)$ *N-homogeneous* $H_n(\mathcal{K}(\mathcal{A},\mathcal{A})) = 0$ *for* $n \geq 1 \Leftrightarrow \mathcal{A}$ *is Koszul.*

- $\mathcal{A}$ Koszul $\Leftrightarrow \mathcal{K}(\mathcal{A},\mathbb{K}) \to \mathbb{K} \to 0$ is a (free) resolution of the trivial left $\mathcal{A}$-module $\mathbb{K}$
- $\mathcal{A}$ Koszul $\Leftrightarrow \mathcal{K}(\mathcal{A},\mathcal{A}) \to \mathcal{A} \to 0$ is a (free) resolution of the $(\mathcal{A},\mathcal{A})$-bimodule $\mathcal{A}$.

$$P_{\mathcal{A}}(t) = \sum_n \text{dim}(\mathcal{A}_n) t^n$$

$$Q_{\mathcal{A}}(t) = \sum_p (\text{dim}(\mathcal{A}^!_{Np}) t^{Np} - \text{dim}(\mathcal{A}^!_{Np+1}) t^{Np+1})$$

$\mathcal{A}$ Koszul $\Rightarrow Q_{\mathcal{A}}(t) P_{\mathcal{A}}(t) = 1$.
which generalizes a well-known result for quadratic algebras since in the latter case ($N=2$) $Q_{\mathcal{A}}(t) = P_{\mathcal{A}^!}(-t)$.

$\boxed{\text{Small complexes } \mathcal{S}(\mathcal{A},\mathcal{M})}$

- $\mathcal{A} = A(E,R) \quad N$-homogeneous
- $\mathcal{M} = (\mathcal{A},\mathcal{A})$-bimodule
- $\mathcal{S}(\mathcal{A},\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}} \mathcal{K}(\mathcal{A},\mathcal{A})$ *small complex*

- If $\mathcal{A}$ is Koszul then the free resolution $\mathcal{K}(\mathcal{A},\mathcal{A}) \to \mathcal{A} \to 0$ of $\mathcal{A} \otimes \mathcal{A}^{\text{opp}}$-modules and the interpretation of the Hochschild homology as

$$H_n(\mathcal{A},\mathcal{M}) = \text{Tor}_n^{\mathcal{A}\otimes\mathcal{A}^{\text{opp}}}(\mathcal{M},\mathcal{A})$$

imply that the small complex $\mathcal{S}(\mathcal{A},\mathcal{M})$ computes the Hochschild homology i.e.

$$H_n(\mathcal{A},\mathcal{M}) = H_n(\mathcal{S}(\mathcal{A},\mathcal{M}))$$

for $n \in \mathbb{N}$.

$\boxed{\text{Dimensions}}$

- $\mathcal{A} = \oplus_{n\in\mathbb{N}}\mathcal{A}_n$ graded algebra
$\mathcal{A}$ has *polynomial growth* if

$$\dim_{\mathbb{K}}(\mathcal{A}_n) \leq Cn^{D-1}, \qquad \forall n \geq 1$$

GK-dim($\mathcal{A}$)= smallest D as above.

- $\mathcal{A}$ N-homogeneous and Koszul
then the above resolutions are minimal projective $\Rightarrow$
global dimension of $\mathcal{A}$ = smallest D
such that $\mathcal{K}_D(\mathcal{A},\mathbb{K}) \neq 0$ and
$\mathcal{K}_n(\mathcal{A},\mathbb{K}) = 0$ for $n > D$.
$\mathcal{A}$ has finite global dimension if
$\mathcal{A}^!_n = 0$ for $n >$ some integer.
Hochschild dim($\mathcal{A}$)= global dim($\mathcal{A}$)
Generically GK-dim($\mathcal{A}$) $\neq$ global dim($\mathcal{A}$).

$\boxed{\text{Gorenstein Homogeneous Algebras}}$

$\mathcal{L}(\mathcal{A}, \mathbb{K})$ = dual of $\mathcal{K}(\mathcal{A}, \mathbb{K})$
$\mathcal{L}(\mathcal{A}, \mathbb{K})$ is a cochain complex of right $\mathcal{A}$-modules (finite, free) and

$$\mathcal{L}(\mathcal{A}, \mathbb{K}) = C_{1,0}(L(\mathcal{A}))$$

$\mathcal{A}$ N-homogeneous and Koszul of finite global dimension $D \Rightarrow \mathcal{L}^n(\mathcal{A}, \mathbb{K}) = 0$ for $n > D$.
Then $\mathcal{A}$ is *Gorenstein* if $\mathcal{L}(\mathcal{A}, \mathbb{K})$ gives a (minimal projective resolution)

$$0 \to \mathcal{L}^0(\mathcal{A}, \mathbb{K}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{L}^D(\mathcal{A}, \mathbb{K}) \to \mathbb{K} \to 0$$

of the trivial right $\mathcal{A}$-module $\mathbb{K}$.
This implies

$$\mathcal{K}_n(\mathcal{A}, \mathbb{K}) \simeq \mathcal{K}_{D-n}(\mathcal{A}, \mathbb{K})$$

$\boxed{\text{Homogeneous Algebra}}$

- Algebras of a monoidal category of vector space $\mathcal{V}$

- Free algebra generated by an object of $\mathcal{V}$ ($\mathcal{V}$ stable by colimits)

- Homogeneous algebras of $\mathcal{V}$

- Duality for N-homogeneous algebra of $\mathcal{V}$ when $\mathcal{V}$ is strict, i.e. when there is an involutive contravariant functor $E \mapsto E^{\#}$ of $\mathcal{V}$, etc.

- Associated N-complexes

X - YANG-MILLS ALGEBRA AND OTHER EXAMPLES

Ref : [13], [26].

Ref : [6], [2].

$\boxed{\text{Yang-Mills Algebra}}$

Yang-Mills algebra = cubic algebra $\mathcal{A}$ generated by ∇_λ, $\lambda \in \{0, \dots, s\}$ with relations

$$g^{\lambda\mu}[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] = 0,\ \nu \in \{0, \dots, s\}$$

$\Rightarrow \mathcal{A} = U(\mathfrak{g})$, $\mathfrak{g} = \sum_{k\geq} \mathfrak{g}_k$ graded Lie algebra

THEOREM 1 *$\mathcal{A}$ is Koszul of global dim $= 3$ and is Gorenstein.*

$\mathcal{A}^!$ generated by θ^λ (dual basis of ∇_λ)

$$\theta^\lambda\theta^\mu\theta^\nu = \frac{1}{s}(g^{\lambda\mu}\theta^\nu + g^{\mu\nu}\theta^\lambda - 2g^{\lambda\nu}\theta^\mu)\mathbf{g}$$

where $\mathbf{g} = g_{\alpha\beta}\theta^\alpha\theta^\beta \in \mathcal{A}^!_2$
$\Rightarrow \mathbf{g}$ central and

$$\mathcal{A}^!_0 = \mathbb{K}\mathbb{1},\ \mathcal{A}^!_1 = \oplus_\lambda \mathbb{K}\theta^\lambda,\ \mathcal{A}^!_2 = \oplus_{\mu,\nu}\mathbb{K}\theta^\mu\theta^\nu$$

$$\mathcal{A}^!_3 = \oplus_\lambda \mathbb{K}\theta^\lambda\mathbf{g},\ \mathcal{A}^!_4 = \mathbb{K}\mathbf{g}^2,\ \mathcal{A}^!_n = 0 \text{ for } n \geq 5$$

$\boxed{\mathcal{K}(\mathcal{A},\mathbb{K}) \text{ for Yang-Mills Algebra}}$

$\mathcal{K}(\mathcal{A},\mathbb{K}) = C_{2,0}$ identifies with

$$0 \longrightarrow \mathcal{A} \overset{\nabla^t}{\underset{d}{\longrightarrow}} \mathcal{A}^{s+1} \overset{M}{\underset{d^2}{\longrightarrow}} \mathcal{A}^{s+1} \overset{\nabla}{\underset{d}{\longrightarrow}} \mathcal{A} \longrightarrow 0$$

where $\nabla = \begin{pmatrix} \nabla_0 \\ \vdots \\ \nabla_s \end{pmatrix}$ and

$$M^{\mu\nu} = (g^{\mu\nu}g^{\alpha\beta} + g^{\mu\alpha}g^{\nu\beta} - 2g^{\mu\beta}g^{\nu\alpha})\nabla_\alpha\nabla_\beta$$

and the arrows mean right matrix multiplication.

Acyclicity in positive degree is straightforward $\Rightarrow$ Koszul of global dim. $= 3$.

Gorenstein follows from sym. by transposition.

$\boxed{\text{Consequences for Yang-Mills Algebra}}$

- $P_{\mathcal{A}}(t) = \frac{1}{(1-t^2)(1-(s+1)t+t^2)}$

$\Rightarrow$ exponential growth for $s \geq 2$

- $P_{\mathcal{A}}(t) = \prod_k \left(\frac{1}{1-t^k}\right)^{\dim(\mathfrak{g}_k)}$ via PBW

- As $\mathcal{A} \otimes \mathcal{A}^{\text{opp}}$-module $\mathcal{A}$ has a free resolution $\mathcal{K}(\mathcal{A}, \mathcal{A}) \to \mathcal{A} \to 0$ which is minimal projective and reads
$0 \to \mathcal{A} \otimes \mathcal{A} \overset{\delta'_3}{\to} \mathcal{A} \otimes \mathbb{K}^{s+1} \otimes \mathcal{A} \overset{\delta'_2}{\to} \mathcal{A} \otimes \mathbb{K}^{s+1} \otimes \mathcal{A} \overset{\delta'_1}{\to}$
$\mathcal{A} \otimes \mathcal{A} \overset{\mu}{\to} \mathcal{A} \to 0$

$\Rightarrow \mathcal{A}$ has Hochschild dimension $= 3$

- Gorenstein property implies here

$$H_n(\mathcal{A}, \mathcal{M}) = H^{3-n}(\mathcal{A}, \mathcal{M})$$

$\boxed{\text{Self-duality Algebra}}$

In the case $s = 3, g_{\mu\nu} = \delta_{\mu\nu}$ i.e. 4-dim. euclidean, the Yang-Mills algebra admits nontrivial quotients $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$.
$\mathcal{A}^{(\varepsilon)}$ ($\varepsilon = \pm$) is generated by ∇_λ ($0 \leq \lambda \leq 3$) with relations

$$[\nabla_0, \nabla_k] = \varepsilon[\nabla_\ell, \nabla_m]$$

$\forall(k, \ell, m)$ cyclic perm. (1,2,3)
$\mathcal{A}^{(+)} \leftrightarrow \mathcal{A}^{(-)}$ by changing orientation $\Rightarrow \mathcal{A}^{(+)}$

THEOREM 2 $\mathcal{A}^{(+)}$ *is a Koszul quadratic algebra of global dimension 2.*

$\mathcal{A}^{(+)!}$ generated by θ^λ with relations

$$\theta^\lambda\theta^\mu + \frac{1}{2}\sum_{\nu,\rho} \epsilon^{\lambda\mu\nu\rho}\theta^\nu\theta^\rho = 0$$

$\Rightarrow \mathcal{A}_0^{(+)!} = \mathbb{K}\mathbb{1},\ \ \mathcal{A}_1^{(+)!} = \oplus_{\lambda=0}^3 \mathbb{K}\theta^\lambda$

$\mathcal{A}_2^{(+)!} = \oplus_{k=1}^3 \mathbb{K}\theta^0\theta^k$ and $\mathcal{A}_n^{(+)!} = 0$ for $n \geq 3$

$$\boxed{\mathcal{K}(\mathcal{A}^{(+)}, \mathbb{K})}$$

$\mathcal{K}(\mathcal{A}^{(+)}, \mathbb{K}) = K(\mathcal{A}^{(+)})$ identifies with

$$0 \longrightarrow \mathcal{A}^{(+)3} \xrightarrow{N} \mathcal{A}^{(+)4} \xrightarrow{\nabla} \mathcal{A}^{(+)} \longrightarrow 0$$

where N is the 3×4-matrix

$$N = \begin{pmatrix} -\nabla_1 & \nabla_0 & \nabla_3 & -\nabla_2 \\ -\nabla_2 & -\nabla_3 & \nabla_0 & \nabla_1 \\ -\nabla_3 & \nabla_2 & -\nabla_1 & \nabla_0 \end{pmatrix}$$

and

$$\nabla = \begin{pmatrix} \nabla_0 \\ \nabla_1 \\ \nabla_2 \\ \nabla_3 \end{pmatrix}$$

Acyclicity in positive degree is straightforward $\Rightarrow$ Theorem.

$\boxed{\text{Consequences for } \mathcal{A}^{(+)}}$

- Free resolution $\mathcal{K}(\mathcal{A}^{(+)}, \mathcal{A}^{(+)}) \to \mathcal{A}^{(+)} \to 0$ of $\mathcal{A}^{(+)}$ as $\mathcal{A}^{(+)} \otimes \mathcal{A}^{(+)\mathrm{opp}}$-module which is minimal.

$\Rightarrow \mathcal{A}^{(+)}$ has Hochschild dimension $= 2$

- $P_{\mathcal{A}^{(+)}}(t) = \frac{1}{(1-t)(1-3t)}$

$\Rightarrow$ exponential growth

- The latter formula also follows from the fact that $\mathcal{A}^{(+)}$ is the universal enveloping algebra of the semi-direct product of the free Lie algebra $L(\nabla_1, \nabla_2, \nabla_3)$ by the derivation δ given by $\delta(\nabla_k) = [\nabla_\ell, \nabla_m]$, $\forall (k, \ell, m) =$ cyclic (1,2,3).

Remembering that $T(\mathbb{K}^3) = \mathbb{K}\langle \nabla_1, \nabla_2, \nabla_3 \rangle$ is of Hochschild dimension 1, this also implies Hochschild-dim$(\mathcal{A}^{(+)}) = 2$

$\boxed{\text{Parafermionic Algebra}}$

$\mathcal{B} = A(E,R)$ cubic algebra
$\dim(E) = D$
$R = \{[[x,y]_\otimes, z]_\otimes | x,y,z \in E\}$
PBW $\Rightarrow$

$$P_\mathcal{B}(t) = \left(\frac{1}{1-t}\right)^D \left(\frac{1}{1-t^2}\right)^{\frac{D(D-1)}{2}}$$

$\Rightarrow gk - \dim(\mathcal{B}) = \frac{D(D+1)}{2}$

$\mathcal{B}^!$ generated by E^* with $\alpha\beta\gamma = \gamma\beta\alpha$, $\theta^3 = 0$, $\forall \alpha, \beta, \gamma, \theta \in E^* \Rightarrow$
$Q_\mathcal{B}(t) = 1 - Dt + \frac{1}{3}D(D^2-1)t^3 - \frac{1}{12}D^2(D^2-1)t^4$
$\Rightarrow P_\mathcal{B} Q_\mathcal{B} = 1 + D(D^2-1)(D^2-4)t^5 F_D(t)$
If $D = 2$ it is an Artin-Schelter algebra
If $D \geq 3$, $F_D(0) \neq 0$ so it is not Koszul

$\boxed{\text{Parabosonic and Plactic Algebras}}$

- $\mathcal{B}$ has a "super" version $\tilde{\mathcal{B}}$ generated by E with relations $[\{x,y\},z] = 0$ $\forall x,y,z \in E$
super PBW $\Rightarrow$

$$P_{\tilde{\mathcal{B}}}(t) = (1+t)^D \left(\frac{1}{1-t^2}\right)^{\frac{D(D+1)}{2}} = P_{\mathcal{B}}(t)$$

- Another classical useful cubic algebra $\mathcal{P}$ has the same Poincaré series, *the plactic algebra*.
$\mathcal{P}$ depends on a basis $(e_i)_{i\in\{1,\dots,D\}}$ of E
it is generated by the e_i with relations

$$e_\ell e_m e_k = e_\ell e_k e_m \text{ for } k < \ell \leq m$$

$$e_k e_m e_\ell = e_m e_k e_\ell \text{ for } k \leq \ell < m$$

- One has $P_{\mathcal{B}}(t) = P_{\tilde{\mathcal{B}}}(t) = P_{\mathcal{P}}(t)$
and $Q_{\mathcal{B}}(t) = Q_{\tilde{B}}(t) = Q_{\mathcal{P}}(t)$

- Same discussion, connection with partitions, multiparametric deformation, etc.