NON-COMMUTATIVE DIFFERENTIAL GEOMETRY AND NEW MODELS OF GAUGE THEORY.

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Abstract. We investigate the non-commutative differential geometry of the algebra $C^\infty(V) \otimes M_n(\mathbb{C})$ of smooth $M_n(\mathbb{C})$-valued functions on a manifold V. For $n \geq 2$, we construct the analogue of Maxwell's theory and interpret it as a field theory on V. It describes a U(n)-Yang-Mills field minimally coupled to a set of fields with values in the adjoint representation which interact among themselves through a quartic polynomial potential. The euclidean action, which is positive, vanishes on exactly two distinct gauge orbits which are interpreted as two vacua of the theory. In one of the corresponding vacuum sectors, the SU(n) part of the Yang-Mills field is massive. For the case n=2, analogies with the standard model of electroweak theory are pointed out. Finally, we briefly describe what happens if one starts from the analogue of a general Yang-Mills theory instead of Maxwell's theory which is a particular case.

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1. INTRODUCTION AND NOTATIONS.

Let V be a smooth manifold and let $C^{\infty}(V)$ be the algebra of smooth complex functions on V considered as an abstract commutative $*$-algebra. Given a smooth complex vector bundle E on V, one denotes by $\Gamma(E)$ the space of smooth sections of E. $\Gamma(E)$ is a finite projective $C^{\infty}(V)$-module. The correspondence $E \to \Gamma(E)$ is an equivalence of the category of smooth complex vector bundles on V with the category of finite projective $C^{\infty}(V)$-modules. There is a notion of connection on finite projective $C^{\infty}(V)$-modules which corresponds to the notion of connection on vector bundles. To define it, it is convenient to use the graded differential algebra $\Omega(V)$ of complex differential forms on V. The Lie algebra of complex vector fields on V can be identified with the Lie algebra $\mathrm{Der}(C^{\infty}(V))$ of derivations of $C^{\infty}(V)$.

In non-commutative differential geometry, the role of $C^{\infty}(V)$ is played by a non-commutative associative algebra $\mathcal{A}$ [1], [2]. Modules of sections of vector bundles are replaced by finite projective $\mathcal{A}$-modules [1], [2]. In order to define connections on $\mathcal{A}$-modules and more generally to define non-commutative generalization of differential calculus, one needs a generalization of differential forms. There are several non-commutative generalizations of the de Rham complex [2], [3], [4]. Here, as in [5], we use as generalization of the algebra of differential forms for $\mathcal{A}$ the graded differential algebra $\Omega_D(\mathcal{A})$ introduced in [4]. We now recall the construction of $\Omega_D(\mathcal{A})$.

Let $\mathrm{Der}(\mathcal{A})$ be the Lie-algebra of derivations of $\mathcal{A}$. This is a generalization of the Lie algebra of vector fields. Recall that a p-cochain ω on the Lie algebra $\mathrm{Der}(\mathcal{A})$ with values in $\mathcal{A}$ is a p-linear antisymmetric mapping of $\mathrm{Der}(\mathcal{A})$ in $\mathcal{A}$, i.e. a linear mapping $\omega : \Lambda^p \mathrm{Der}(\mathcal{A}) \to \mathcal{A}$. The space of p-cochains of $\mathrm{Der}(\mathcal{A})$ with values in $\mathcal{A}$ is denoted by $C^p(\mathrm{Der}(\mathcal{A}),\mathcal{A})$. The direct sum $C(\mathrm{Der}(\mathcal{A}),\mathcal{A}) = \bigoplus_{p\in\mathbb{N}} C^p(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ is naturally a graded algebra. It is a graded differential algebra with differential d defined by

$$d\omega(X_0,X_1,\dots,X_p) = \sum_{0\leq k\leq p} (-1)^k X_k \omega(X_0,\overset{k}{\check{\dots}},X_p) + \sum_{0\leq r<s\leq p} (-1)^{r+s} \omega([X_r,X_s], X_0,\overset{r\ s}{\check{\dots}\check{\dots}},X_p)$$

for $\omega \in C^p(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ and $X_0,X_1,\dots,X_p \in \mathrm{Der}(\mathcal{A})$. One has $\mathcal{A} = C^0(\mathrm{Der}(\mathcal{A}),\mathcal{A}) \subset C(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ and the graded differential algebra $\Omega_D(\mathcal{A})$ is defined to be the smallest differential sub-algebra of $C(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ which contains $\mathcal{A}$. Any element of $\Omega_D^p(\mathcal{A})$ is a sum of elements of the form $A_0 dA_1 \dots dA_p$ with $A_0,A_1,\dots,A_p \in \mathcal{A}$.

$\Omega_D(C^\infty(V))$ coincides with the graded differential algebra $\Omega(V)$ of differential forms on V.

In this paper, we investigate the non-commutative differential geometry of the algebra $\mathcal{A} = C^\infty(V)\otimes M_n(\mathbb{C})$ of smooth $M_n(\mathbb{C})$-valued function on a connected, simply-connected manifold V . Some aspects of the non-commutative geoemtry of algebras of that type were investigated in [6] in a different context. We use $\Omega_D(\mathcal{A})$ as the analogue of the differential algebra of exterior forms. We show in Section 2 that $\Omega_D(\mathcal{A}) = \Omega_D(C^\infty(V))\otimes\Omega_D(M_n(\mathbb{C}))$. The second factor $\Omega_D(M_n(\mathbb{C}))$ was investigated in [5] . We introduce in Section 3 the analogue of a metric for $\mathcal{A}$ and the corresponding scalar product on $\Omega_D(\mathcal{A})$. In Section 4 , we study connections on the free hermitian $\mathcal{A}$-modules. It is shown that, for $n \geqslant 2$, there are several gauge orbits of flat connections .

In Section 5, V is the (s+1)-dimensional euclidean space-time $\mathbb{R}^{s+1}$ and we describe the analogue for $\mathcal{A}$ of the Maxwell action. This is an action for connections on the free hermitian $\mathcal{A}$-module of rank one. We interpret the corresponding theory in terms of a field theory on space-time. It consists of a U(n)-Yang-Mills field minimally coupled to a set of scalar fields with value in the adjoint representation which interact among themselves through a quartic polynomial potential. The euclidean action, which is positive, vanishes on two distinct gauge orbits. These are interpreted as two vacua for the corresponding quantum field theory. In one of the corresponding vacuum sectors, the SU(n) part of the Yang-Mills field is massive. This sector is the most natural one from the point of view of the non-commutative geometry of $\mathcal{A}$ since the vacuum there corresponds to the pure gauge connections i.e. the pure gauge non-commutative Maxwell potentials. For the case $n=2$ we discuss the analogies and the differences with the standard model of the electroweak interactions, see for example [7] . Finally, we describe the analogue for $\mathcal{A}$ of the U(r)-Yang-Mills action. It is an action for connections on the free hermitian $\mathcal{A}$-module of rank r . In Section 6 we present our conclusions.

2. DIFFERENTIAL CALCULUS FOR $\mathcal{A} = C^{\infty}(V) \otimes M_n(\mathbb{C})$.

2.1 <u>The Lie algebra</u> Der($\mathcal{A}$). $\mathcal{A} = C^{\infty}(V) \otimes M_n(\mathbb{C})$ and $M_n(\mathbb{C})$ are naturally *-algebras with units. Associated with any point $x \in V$, there is a homomorphism $\gamma_x : \mathcal{A} \rightarrow M_n(\mathbb{C})$ of *-algebras with units defined by

$\gamma_x(f\otimes M) = f(x)M$, $\forall f \in C^\infty(V)$ and $\forall M \in M_n(\mathbb{C})$. This γ_x is the evaluation at $x \in V$. The subalgebra $C^\infty(V) \otimes \mathbf{1}$ of $\mathcal{A}$ is the center of $\mathcal{A}$. The Lie algebra $\mathrm{Der}(\mathcal{A})$ of all derivations of $\mathcal{A}$ is a module over the center $C^\infty(V)\otimes\mathbf{1}$ of $\mathcal{A}$, so it is a $C^\infty(V)$-module. $\mathrm{Der}(C^\infty(V))$ is the Lie algebra of smooth vector fields on V and $\mathrm{Der}(M_n(\mathbb{C}))$ is the Lie algebra $\mathfrak{sl}(n,c)$ [4, 5]. It is clear that $(\mathrm{Der}(C^\infty(V)) \otimes \mathbf{1}) \oplus (C^\infty(V) \otimes \mathrm{Der}(M_n(\mathbb{C})))$ is a Lie sub-algebra and a $C^\infty(V)$-submodule of $\mathrm{Der}(\mathcal{A})$. It is in fact $\mathrm{Der}(\mathcal{A})$.

2.2 LEMMA. One has $\mathrm{Der}(\mathcal{A}) = (\mathrm{Der}(C^\infty(V)) \otimes \mathbf{1}) \oplus (C^\infty(V) \otimes \mathrm{Der}(M_n(\mathbb{C})))$

Proof. Let X be a derivation of $\mathcal{A}$. Then $f \mapsto X(f\otimes\mathbf{1})$ is a $M_n(\mathbb{C})$-valued vector field on V. One has $X(f\otimes M) = X((f\otimes\mathbf{1})(\mathbf{1}\otimes M)) = X((\mathbf{1}\otimes M)(f\otimes\mathbf{1}))$, i.e. $X(f\otimes\mathbf{1})\, \mathbf{1}\otimes A + f\otimes\mathbf{1}\, X(\mathbf{1}\otimes A) = \mathbf{1}\otimes A\ X(f\otimes\mathbf{1}) + X(\mathbf{1}\otimes A)\, f\otimes\mathbf{1}$ and therefore $X(f\otimes\mathbf{1})\, \mathbf{1}\otimes A = \mathbf{1}\otimes A\, X(f\otimes\mathbf{1})$, $\forall f \in C^\infty(V)$, $\forall M \in M_n(\mathbb{C})$. It follows that $X(f\otimes\mathbf{1})$ is in $C^\infty(V)\otimes\mathbf{1}$, $\forall f \in C^\infty(V)$. This shows that the restriction $X \restriction C^\infty(V)\otimes\mathbf{1}$ is in $\mathrm{Der}(C^\infty(V))\otimes\mathbf{1}$. The mapping $M \mapsto \gamma_x(X(\mathbf{1}\otimes M))$ is a derivation of $M_n(\mathbb{C})$, $\forall x \in V$. This implies that the restriction $X \restriction \mathbf{1}\otimes M_n(\mathbb{C})$ is in $C^\infty(V)\otimes\mathrm{Der}(M_n(\mathbb{C}))$. $\square$

2.3 The graded differential algebra $\Omega_D(\mathcal{A})$. We recall that if Ω_0 and Ω_1 are graded differential algebras with differentials d_0 and d_1 then $\Omega_0\otimes\Omega_1$ is naturally a graded differential algebra if one defines the product by $(x\otimes y)(z\otimes t) = (-1)^{rs}\, xz\otimes yt$ for $x \in \Omega_0$, $y \in \Omega_1^r$, $z \in \Omega_0^s$, $t \in \Omega_1$ and the differential d by $d(x\otimes y) = d_0 x\otimes y + (-1)^p\, x\otimes d_1 y$, $\forall x \in \Omega_0^p$, $\forall y \in \Omega_1$.

It follows from lemma 2.2 that

$$C(\mathrm{Der}(C^\infty(V)), C^\infty(V))\otimes C(\mathrm{Der}(M_n(\mathbb{C})), M_n(\mathbb{C}))$$

is a graded differential subalgebra of $C(\mathrm{Der}(\mathcal{A}),\mathcal{A})$. On the other hand, $\Omega_D(C^\infty(V))$ is the smallest differential subalgebra of $C(\mathrm{Der}(C^\infty(V)), C^\infty(V))$ which contains $C^\infty(V)$ and $\Omega_D(M_n(\mathbb{C}))$ is the smallest differential subalgebra of $C(\mathrm{Der}\, M_n(\mathbb{C}), M_n(\mathbb{C}))$ which contains $M_n(\mathbb{C})$. Therefore the smallest differential subalgebra $\Omega_D(\mathcal{A})$ of $C(\mathrm{Der}(\mathcal{A}),\mathcal{A})$ which contains $\mathcal{A} = C^\infty(V)\otimes M_n(\mathbb{C})$ is $\Omega_D(C^\infty(V))\otimes\Omega_D(M_n(\mathbb{C}))$. Thus one has:

$$\Omega_D(\mathcal{A}) = \Omega_D(C^\infty(V))\otimes\Omega_D(M_n(\mathbb{C})).$$

In fact, [4], $\Omega_D(C^\infty(V))$ is the graded differential algebra $\Omega(V)$ of exterior differential forms on V and $\Omega_D(M_n(\mathbb{C}))$ coincides with $C(\mathrm{Der}(M_n(\mathbb{C})), M_n(\mathbb{C})) = M_n(\mathbb{C})\otimes\wedge\mathfrak{sl}(n,\mathbb{C})^*$, [4, 5]. So one has:

$$\Omega_D(\mathcal{A}) = \Omega(V)\otimes M_n(\mathbb{C})\otimes\wedge\mathfrak{sl}(n,\mathbb{C})^*.$$

2.4 <u>Remark</u>. For algebras $\mathcal{B}$ and $\mathcal{C}$, $\Omega_D(\mathcal{B}\otimes\mathcal{C})$ is generally distinct from $\Omega_D(\mathcal{B})\otimes\Omega_D(\mathcal{C})$. For instance $\Omega_D(M_r(\mathbb{C})) = M_r(\mathbb{C})\otimes\wedge\mathfrak{sl}(r,\mathbb{C})^*$, $\Omega_D(M_s(\mathbb{C})) = M_s(\mathbb{C})\otimes\wedge\mathfrak{sl}(s,\mathbb{C})^*$ and $\Omega_D(M_r(\mathbb{C})\otimes M_s(\mathbb{C})) = M_r(\mathbb{C})\otimes M_s(\mathbb{C})\otimes\wedge\mathfrak{sl}(rs,\mathbb{C})^*$. In fact one has: $\mathfrak{sl}(rs,\mathbb{C}) = (\mathfrak{sl}(r,\mathbb{C})\otimes\mathfrak{sl}(s,\mathbb{C}))\oplus(\mathfrak{sl}(r,\mathbb{C})\otimes\mathbf{1})\oplus(\mathbf{1}\otimes\mathfrak{sl}(s,\mathbb{C}))$.

2.5 $\Omega_D(\mathcal{A})$ <u>as bigraded differential algebra</u>. $\Omega_D(\mathcal{A})$ is naturally a bigraded algebra if one sets $\Omega_D^{r,s}(\mathcal{A}) = \Omega^r(V)\otimes\Omega_D^s(M_n(\mathbb{C}))$. We identify $\Omega(V)$, (resp. $\Omega_D(M_n(\mathbb{C}))$), with the differential subalgebra $\Omega(V)\otimes\mathbf{1}$, (resp. $\mathbf{1}\otimes\Omega_D(M_n(\mathbb{C}))$), of $\Omega_D(\mathcal{A})$. We denote by d the differential of $\Omega_D(\mathcal{A})$. Let d' be the unique antiderivation of $\Omega_D(\mathcal{A})$ extending the exterior differential of $\Omega(V)$ such that $d'\Omega_D(M_n(\mathbb{C})) = 0$ and let d" be the unique antiderivation of $\Omega_D(\mathcal{A})$ extending the differential of $\Omega_D(M_n(\mathbb{C}))$ such that $d''\Omega(V) = 0$. Then d' is of bidegree (1,0), d" is of bidegree (0,1) and one has: $d = d' + d''$, $d'^2 = d''^2 = d''d' + d'd'' = 0$. In other words, $\Omega_D(\mathcal{A})$ is a bigraded differential algebra.

2.6 <u>Reality</u>. $\mathcal{A}$ is a $*$-algebra. We denote by $\mathrm{Der}_{\mathbb{R}}(\mathcal{A})$ the real Lie subalgebra of $\mathrm{Der}(\mathcal{A})$ of derivations X such that $X(A^*) = (X\,A)^*$. One has $\mathrm{Der}_{\mathbb{R}}(\mathcal{A}) = (\mathrm{Der}_{\mathbb{R}}(C^\infty(V))\otimes\mathbf{1})\oplus(C^\infty_{\mathbb{R}}(V)\otimes\mathrm{Der}_{\mathbb{R}}(M_n(\mathbb{C})))$ where $\mathrm{Der}_{\mathbb{R}}(C^\infty(V))$ is the real Lie algebra of real vector fields on V, $\mathrm{Der}_{\mathbb{R}}(M_n(\mathbb{C}))$ is the Lie algebra $\mathfrak{su}(n)$ for its adjoint action on $M_n(\mathbb{C})$ and $C^\infty_{\mathbb{R}}(V)$ is the real algebra of real functions on V. Correspondingly, there is an antilinear involution $\omega \mapsto \bar{\omega}$ on $\Omega_D(\mathcal{A})$ which extends the involution of $\mathcal{A}$. One has $\overline{\alpha\otimes\alpha'} = \bar{\alpha}\otimes\bar{\alpha}'$ for $\alpha \in \Omega(V)$ and $\alpha' \in \Omega_D(M_n(\mathbb{C}))$ where $\alpha \mapsto \bar{\alpha}$ is the complex conjugaison of differential forms on V and $\alpha' \mapsto \bar{\alpha}'$ is the involution of $\Omega_D(M_n(\mathbb{C}))$ defined in [5]. An element ω of $\Omega_D(\mathcal{A})$ will be said to be real, (resp. imaginary), if $\bar{\omega} = \omega$, (resp. $\bar{\omega} = -\omega$).

3. METRIC FOR $\mathcal{A}$ AND SCALAR PRODUCT ON $\Omega_D(\mathcal{A})$.

3.1 Basis for $M_n(\mathbb{C})$ and expressions for d''. For the differential calculus of $M_n(\mathbb{C})$ (or $\Omega_D(M_n(\mathbb{C}))$) we use the notations of [5], except that, since we consider $\Omega_D(M_n(\mathbb{C}))$ as embedded in $\Omega_D(\mathcal{A})$, the differential of $\Omega_D(M_n(\mathbb{C}))$ will be denoted d'' (or d) to be consistent with 2.5. We shall use a basis $E_k, k \in \{1,\dots,n^2-1\}$ of hermitian traceless nxn matrices which is orthonormal in the sense that $\frac{1}{n}\,\mathrm{Tr}\,(E_k E_\ell) = \delta_{k\ell}$. So one has a multiplication table in $M_n(\mathbb{C})$ of the form:

$$E_k E_\ell = \delta_{k\ell}\,\mathbf{1} + \sum_m (s_{k\ell m} - \frac{i}{2} C_{k\ell m})\, E_m \qquad (1)$$

with $s_{k\ell m} = s_{\ell k m} \in \mathbb{R}$ and $C_{k\ell m} = - C_{\ell k m} \in \mathbb{R}$. Associativity then implies that $s_{k\ell m}$ is completely symmetric, that $C_{k\ell m}$ is completely antisymmetric and that they satisfy some relations (See [8] for instance).

It follows from these relations that one has $\sum_{k,\ell} C_{k\ell r}\, C_{k\ell s} = 2\, n^2 \delta_{rs}$. Let us introduce the basis $\partial_k = \mathrm{ad}(iE_k)$ of $\mathrm{Der}_{\mathbb{R}}(M_n(\mathbb{C})) = \mathfrak{su}(n)$. One has

$[\partial_k,\partial_\ell] = \sum_m C_{k\ell m}\partial_m$. Define $\theta^k \in \Omega^1_D(M_n(\mathbb{C})) \subset \Omega_D(\mathcal{A})$, $k \in \{1,\dots,n^2-1\}$, by

$\theta^k(\partial_\ell) = \delta^k{}_\ell \mathbf{1}$ for $k,\ell \in \{1,\dots,n^2-1\}$. One has in $\Omega_D(\mathcal{A})$:

$$A\theta^k = \theta^k A,\ \forall A \in \mathcal{A} \quad (2) \qquad \text{and} \quad \theta^k\theta^\ell = -\theta^\ell\theta^k \quad (3).$$

The differential d'' is then characterized by:

$$d''\alpha = 0,\ \forall \alpha \in \Omega(V) \quad (4) \qquad d''E_k = -\sum_{m,\ell} C_{k\ell m}\, E_m\, \theta^\ell \quad (5)$$

and $d''\theta^k = -\frac{1}{2}\sum_{\ell,m} C_{k\ell m}\,\theta^\ell\theta^m$ (6) . By introducing the canonical element θ of $\Omega^1_D(M_n(\mathbb{C}))$, [5], defined by $\theta = E_k\theta^k$ and using (4), (5) can be rewritten in the form:

$$d''A = i\,[\theta, A],\quad \forall A \in \mathcal{A} \qquad (5')$$

The relation (6) may be inverted to yield $\theta^k = -\frac{i}{n^2}\sum_\ell E_\ell\, E_k\, d''E_\ell$.

The differential d' is characterized by the fact that it coincides with the exterior differential on $\Omega(V) \subset \Omega_D(\mathcal{A})$ and that it satisfies $d'E_k = 0$ and $d'\theta^k = 0$ for $k \in \{1,\dots,n^2-1\}$.

3.2 Metric for $\mathcal{A}$. We now assume that V is an oriented riemannian manifold with metric ds^2. In local coordinates (x^μ), $ds^2 = g_{\mu\nu}dx^\mu \otimes dx^\nu$ and $(g^{\mu\nu})$ will denote the inverse matrix of $(g_{\mu\nu})$.

In [5] we introduced what we called there the canonical riemannian structure for $M_n(\mathbb{C})$ which becomes with conventions adopted here

$\sum_k \theta^k \otimes \theta^k$. It is natural to combine these structures by introducing the

metric $ds^2 + \left(\frac{1}{m}\right)^2 \sum_k \theta^k \otimes \theta^k$ for $\mathcal{A}$, where $\frac{1}{m}$ is a positive constant.

We have in mind the case where $V = \mathbb{R}^{s+1}$ is the $(s+1)$-dimensional euclidean space time and where $ds^2 = \sum_\mu dx^\mu \otimes dx^\mu$ has the dimension of the square of a length. In this case $\frac{1}{m}$ is a length, i.e. m is a mass in standard units where $\hbar = c = 1$.

3.3 <u>Scalar product for</u> $\Omega_D(\mathcal{A})$. Associated with the metric and the orientation of V , there is the star isomorphism $\alpha \mapsto *\alpha$ of $\Omega(V)$ and the corresponding positive hermitian scalar product on $\Omega(V)$ such that $\langle \alpha|\beta \rangle = \int_V \bar{\alpha} \wedge *\beta$ for $\alpha, \beta \in \Omega^p(V)$ and $\langle \alpha|\beta \rangle = 0$ if α and β have different degrees. Strictly speaking, this scalar product is defined on $\Omega(V)$ only if V is compact. Otherwise one has to restrict attention to forms for which $\langle \alpha|\alpha \rangle < \infty$, for instance to forms with compact support. However we shall not be concerned with this here since the scalar product will be used only to write formal lagrangians for euclidean field theories.

In [5] we constructed a star isomorphism of $\Omega_D(M_n(\mathbb{C}))$ associated to $\sum_k \theta^k \otimes \theta^k$ and then defined a scalar product on $\Omega_D(M_n(\mathbb{C}))$ by using this star isomorphism and a generalization of integration (essentially the trace). The only thing that the rescaling $\sum_k \theta^k \otimes \theta^k \mapsto \left(\frac{1}{m}\right)^2 \sum_k \theta^k \otimes \theta^k$ changes is the scalar product $\langle \alpha'' | \beta'' \rangle$ of $\alpha'', \beta'' \in \Omega_D^p(M_n(\mathbb{C}))$. It becomes $\left(\frac{1}{m}\right)^{n^2-1} \times m^{2p}$-times the scalar product of [5] which corresponds to the case $m=1$.

We now define a scalar product $\langle . | . \rangle$ on $\Omega_D(\mathcal{A})$ by
$\langle \alpha' \otimes \alpha'' | \beta' \otimes \beta'' \rangle = \langle \alpha' | \beta' \rangle \langle \alpha'' | \beta'' \rangle, \quad \forall \alpha', \beta' \in \Omega(V), \quad \forall \alpha'', \beta'' \in \Omega_D(M_n(\mathbb{C}))$.

This is just the scalar product we would obtain from

$ds^2 + \left(\frac{1}{m}\right)^2 \sum_k \theta^k \otimes \theta^k$ by proceeding as in [5].

4. CONNECTIONS ON HERMITIAN $\mathcal{A}$-MODULES.

4.1 <u>Hermitian</u> $\mathcal{A}$-<u>modules</u>. An element P of $\mathcal{A}$ is <u>positive</u> if $P = A^*A$ for some $A \in \mathcal{A}$. The set $\mathcal{A}^+$ of positive element of $\mathcal{A}$ is a convex cone in $\mathcal{A}$. Let $\mathcal{M}$ be a right $\mathcal{A}$-module. A hermitian structure on $\mathcal{M}$ is a $\mathcal{A}$-valued positive definite hermitian form on $\mathcal{M}$, $(\Psi, \Phi) \mapsto h(\Psi, \Phi) \in \mathcal{A}$

$(\Psi,\Phi \in \mathcal{M})$, such that one has $h(\Psi A,\Phi B) = A^* h(\Psi,\Phi)B$, $\forall\Psi,\Phi \in \mathcal{M}$, $\forall A,B \in \mathcal{A}$. Positive definite means that $h(\Psi,\Psi) \in \mathcal{A}^+$, $\forall\Psi \in \mathcal{M}$, and that $h(\Psi,\Psi) = 0$ implies $\Psi = 0$.

A right $\mathcal{A}$-module equipped with a hermitian structure will be called a hermitian $\mathcal{A}$-module.

$\mathcal{A}^r$ is naturally a right $\mathcal{A}$-module : $(A_1,\dots A_r)A = (A_1 A,\dots,A_r A)$ $\forall(A_1,\dots,A_r) \in \mathcal{A}^r$, $\forall A \in \mathcal{A}$. It is a hermitian $\mathcal{A}$-module if one defines its hermitian structure by $h((A_1,\dots,A_r),(B_1,\dots,B_r)) = \sum_{a=1}^{a=r} A^*_a B_a$.

Conversely, let $\mathcal{H}^r$ be a free hermitian $\mathcal{A}$-module of rank r with hermitian structure h. Then one can construct an orthonormal basis (e_a) $a \in \{1,\dots r\}$ of $\mathcal{H}^r$, i.e. $e_a \in \mathcal{H}^r$ such that $h(e_a,e_b) = \delta_{ab}\mathbb{1}$ $\forall a,b \in \{1,\dots,r\}$. We shall call such an orthonormal basis <u>a gauge</u>. Given such a gauge, $\Psi \in \mathcal{H}^r$ can be written $\Psi = \sum_a e_a A_a$ in a unique way with $A_a \in \mathcal{A}$.

Furthermore if $\Phi = \sum e_a B_a$ is another element of $\mathcal{H}^r$, then $h(\Psi,\Phi) = \sum_a A^*_a B_a$. Thus each gauge gives an isomorphism $\mathcal{H}^r \to \mathcal{A}^r$ of hermitian $\mathcal{A}$-modules. A change of orthonormal basis will be called a <u>gauge transformation</u>. Such a gauge transformation U is a unitary element of $\mathcal{A}\otimes M_r(\mathbb{C}) = C^\infty(V)\otimes M_n(\mathbb{C})\otimes M_r(\mathbb{C}) = C^\infty(V)\otimes M_{nr}(\mathbb{C})$. So U is a $U(nr)$-valued function on V

4.2 <u>Connections</u>. Let $\mathcal{M}$ be a right $\mathcal{A}$-module. A Ω_D-<u>connection</u> or simply a <u>connection</u> on $\mathcal{M}$ [2] is a linear mapping $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A})$ such that $\nabla(\Phi A) = (\nabla\Phi)A + \Phi \otimes dA$, $\forall \Phi \in \mathcal{M}$, $\forall A \in \mathcal{A}$. If $\mathcal{M}$ is a hermitian $\mathcal{A}$-module with hermitian structure h, ∇ will be called a <u>hermitian connection</u> if it satisfies $d\,h(\Phi,\Psi) = h(\nabla\Phi,\Psi) + h(\Phi,\nabla\Psi)$, $\forall \Phi, \Psi \in \mathcal{M}$.

Connections always exist on projective modules of finite type [2].

Let ∇ be a connection on $\mathcal{M}$. One extends ∇ as a linear mapping, again denoted by ∇, of $\mathcal{M} \otimes_{\mathcal{A}} \Omega_D(\mathcal{A})$ in itself setting [2] : $\nabla(\Phi \otimes \alpha) = (\nabla\Phi)\alpha + \Phi \otimes d\alpha$, $\forall \Phi \in \mathcal{M}$, $\forall \alpha \in \Omega_D(\mathcal{A})$. Consider $\nabla^2 : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A})$. One has $\nabla^2(\Phi A) = (\nabla^2\Phi)A$, $\forall \Phi \in \mathcal{M}$, $\forall A \in \mathcal{A}$. Thus ∇^2 is a right $\mathcal{A}$-module homomorphism which is <u>the curvature of</u> ∇.

4.3 <u>Connections on the free hermitian $\mathcal{A}$-module of rank</u> r. We consider $\mathcal{A}^r$ as a hermitian right $\mathcal{A}$-module as explained in 4.1. The canonical basis of $\mathcal{A}^r$ will be denoted by $e = (e_1,\dots,e_r)$. We denote by $\mathcal{U}_r$ the group of gauge transformations, i.e. the group of unitary elements of $M_r(\mathcal{A}) = \mathcal{A} \otimes M_r(\mathbb{C})$. Any othonormal basis, or gauge, of $\mathcal{A}^r$ is of the form eU for a unique $U \in \mathcal{U}_r$.

Let ∇ be a connection on $\mathcal{A}^r$. Then $\nabla e_a = e_b \otimes \omega^b_a$ for $a \in \{1,\dots,r\}$, $\omega^a_b \in \Omega^1_D(\mathcal{A})$. Furthermore, ∇ is hermitian if and only if $\bar{\omega}^a_b = -\omega^b_a$. We write the relations $\nabla e_a = e_b \omega^b_a$ in the form $\nabla e = e\omega$ with $\omega = (\omega^a_b) \in M_r(\Omega^1_D(\mathcal{A})) = \Omega^1_D(\mathcal{A}) \otimes M_r(\mathbb{C})$. The element ω of $\Omega^1_D(\mathcal{A}) \otimes M_r(\mathbb{C})$ will be called <u>the component of</u> ∇ <u>in</u> e or simply the component of ∇. Each $\omega \in \Omega^1_D(\mathcal{A}) \otimes M_r(\mathbb{C})$ is the component in e of a unique connection ∇. We could define similarily the component of ∇ in an arbitrary gauge eU: If ω is its component in e, then its component in eU is $U^{-1}\omega U + U^{-1}dU$. Here however, we consider $U^{-1}\omega U + U^{-1}dU$ as the component in e of another connection denoted ∇^U. $\nabla \mapsto \nabla^U$, $U \in \mathcal{U}_r$ is a right action of the gauge group $\mathcal{U}_r$ on the space of connections on $\mathcal{A}^r$. ∇^U is hermitian if and only if ∇ is hermitian. The set $\{\nabla^U | U \in \mathcal{U}_r\}$ will be called <u>the gauge orbit of</u> ∇. In the same way, $\nabla^2 e = e\varphi$ with $\varphi \in \Omega^2_D(\mathcal{A}) \otimes M_r(\mathbb{C})$. One has $\varphi = d\omega + \omega^2$ in the algebra $\Omega_D(\mathcal{A}) \otimes M_r(\mathbb{C})$ where d is defined by $d(\alpha \otimes x) = d\alpha \otimes x$, $\forall \alpha \in \Omega_D(\mathcal{A})$, $\forall x \in M_r(\mathbb{C})$. φ will be called <u>the component of the curvature</u> ∇^2 of ∇. If φ is the component of ∇^2 then the component of the curvature $(\nabla^U)^2$ of ∇^U is $U^{-1}\varphi U$.

4.4 <u>Flat hermitian connections on</u> $\mathcal{A}^r$. A connection is called a flat connection if its curvature vanishes. Thus a connection ∇ on $\mathcal{A}^r$ with component ω is flat if and only if $d\omega + \omega^2 = 0$. If $U \in \mathcal{U}_r$, then ∇^U is flat if and only if ∇ is flat.

For each gauge eU^{-1} $(U \in \mathcal{U}_r)$ there is a unique connection $\overset{(eU^{-1})}{\nabla}$ such that $\overset{(eU^{-1})}{\nabla}(eU^{-1}) = 0$. Its component in e is $U^{-1}dU$ so one has $\overset{(eU^{-1})}{\nabla} = \overset{(e)}{\nabla}{}^{U}$ and it is a flat hermitian connection. These connections $\overset{(e)}{\nabla}{}^{U}$, $U \in \mathcal{U}_r$, will be called <u>pure gauge connections</u>. The set of pure gauge connections is a gauge orbit of flat hermitian connections on $\mathcal{A}^r$. In the commutative case where $\mathcal{A} = C^\infty(V)$ they are the only flat hermitian connections on $\mathcal{A}^r$. However for $\mathcal{A} = C^\infty(V) \otimes M_n(\mathbb{C})$ with $n \geqslant 2.$, there are other gauge orbits of hermitian flat connections on $\mathcal{A}^r$ which we now describe.

We now assume that $\mathcal{A} = C^\infty(V) \otimes M_n(\mathbb{C})$ with $n \geqslant 2$ and we let r denote a positive integer with $r \geqslant 1$. Let $\overset{\alpha}{R}_k$, $k \in \{1,2,\dots,n^2-1\}$, $\alpha \in \{0,1,\dots,N(n,r)\}$ be a set of antihermitian elements of $M_n(\mathbb{C}) \otimes M_r(\mathbb{C})$ such that $\overset{0}{R}_k = 0$, $\overset{1}{R}_k = iE_k \otimes \mathbf{1}$, $[\overset{\alpha}{R}_k, \overset{\alpha}{R}_\ell] = \sum_m C_{k\ell m} \overset{\alpha}{R}_m$ (i.e. $\overset{\alpha}{R}$ is a representation of $\mathfrak{su}(n)$ in $\mathbb{C}^n \otimes \mathbb{C}^r$), $\forall \alpha, k, \ell$, and such that, if (R_k) are n^2-1 antithermitian elements of $M_n(\mathbb{C}) \otimes M_r(\mathbb{C})$ satisfying $[R_k, R_\ell] =$

$\sum_m c_{k\ell m} R_m$, $\forall k,\ell$, then there is a unique $\alpha \in \{0,1\ldots,N(n,r)\}$ and a unitary $V \in M_n(\mathbb{C})\otimes M_r(\mathbb{C})$ such that $R_k = V^{-1}\overset{\alpha}{R}_k V$, $\forall k$.

In other words $(\overset{\alpha}{R})$ is a complete set of multually inequivalent antihermitian representations of $\mathfrak{su}(n)$ in $\mathbb{C}^n\otimes\mathbb{C}^r$. Let $\overset{\alpha}{\nabla}$ be the connection on $\mathcal{A}^r$ with component $(\overset{\alpha}{R}_k - iE_k\otimes\mathbb{1})\theta^k \in \Omega^1_D(\mathcal{A})\otimes M_r(\mathbb{C})$, $\forall\alpha \in \{0,1,\ldots,N(n,r)\}$. The $\overset{\alpha}{\nabla}$ are hermitian connections and one has the following result.

4.5 <u>THEOREM</u>. a) <u>The</u> $\overset{\alpha}{\nabla}$ <u>are flat hermitian connections and, if</u> α <u>is distinct of</u> β, <u>the gauge orbits of</u> $\overset{\alpha}{\nabla}$ <u>and of</u> $\overset{\beta}{\nabla}$ <u>are distinct</u>.

b) <u>A hermitian connection</u> ∇ <u>on</u> $\mathcal{A}^r$ <u>is flat if and only if it is an element of the gauge orbit of</u> $\overset{\alpha}{\nabla}$ <u>for some</u> $\alpha \in \{0,1\ldots,N(n,r)\}$, <u>i.e.</u> $\nabla = \overset{\alpha}{\nabla}{}^U$ <u>with</u> $U \in \mathcal{U}_r$ <u>and</u> $\alpha \in \{0,\ldots,N(n,r)\}$.

<u>Proof</u>. Let ∇ be a hermitian connection on $\mathcal{A}^r$ with component ω. Write ω in the form $\omega = A + (B_k - iE_k\otimes\mathbb{1})\theta^k$ where A is a one-form on V with values in the antihermitian elements of $M_n(\mathbb{C})\otimes M_r(\mathbb{C})$ and where the B_k are functions on V with values in the antihermitian elements of $M_n(\mathbb{C})\otimes M_r(\mathbb{C})$. One has:

$$d\omega + \omega^2 = d'A + A^2 + (d'B_k + [A,B_k])\theta^k + \frac{1}{2}([B_k,B_\ell] - \sum_m c_{k\ell m} B_m)\theta^k\theta^\ell \tag{7}$$

∇ is flat if and only if $d'A + A^2 = 0$, $d'B_k + [A,B_k] = 0$, $\forall k$, and $[B_k,B_\ell] = \sum_m c_{k\ell m} B_m$, $\forall k,\ell$. It follows that the $\overset{\alpha}{\nabla}$ are flat connections. If $\overset{\beta}{\nabla} = \overset{\alpha}{\nabla}{}^U$, U may be chosen to be constant and then $\overset{\beta}{R}_k = U^{-1} \overset{\alpha}{R}_k U$ which is in contradiction with the assumptions on the R^α.

Suppose that ∇ is a flat hermitian connection. Then $d'A + A^2 = 0$ implies $A = U^{-1}d'U$ and $[B_k,B_\ell] = \sum_m c_{k\ell m} B_m$ implies $B_k = V^{-1}\overset{\alpha}{R}_k V$ for some $\alpha \in \{0,1,\ldots,N(n,r)\}$ and $U,V \in \mathcal{U}_r$. Furthermore $d'B_k + [A,B_k] = 0$, implies $d'(UV^{-1}\overset{\alpha}{R}_k VU^{-1}) = 0$ so one can choose U and V such that $U = V$. This implies $\nabla = \overset{\alpha}{\nabla}{}^U$. $\square$

4.6 <u>Remarks</u>. a) Under a gauge transformation $\nabla \mapsto \nabla^U$, $U \in \mathcal{U}_r$, A and B_k as above transform as $A \mapsto U^{-1}AU + U^{-1}d'U$ and $B_k \mapsto U^{-1}B_kU$. Thus the B_k transform homogeneously. This is in fact the reason why we represent the component ω of ∇ in the form $\omega = A + (B_k - iE_k \otimes \mathbb{1})\theta^k$ and why we introduce the component of $\overset{\alpha}{\nabla}$ in the form $(R^{\alpha}_{k} - iE_{k}\otimes\mathbb{1})\theta^{k}$. It is connected with what was described in [5] lemma 7.3 for matrix algebras.

b) One has $\overset{1}{\nabla} = \overset{(e)}{\nabla}$ so the pure gauge connections on $\mathcal{A}^{r}$ are the elements of the gauge orbit of $\overset{1}{\nabla}$.

c) For any $r \geqslant 1$, $N(n,r) \geqslant 1$ $(n \geqslant 2)$ so one has at least two gauge orbits of flat hermitian connections on $\mathcal{A}^{r}$: The orbit of $\overset{0}{\nabla}$ and the orbit of $\overset{1}{\nabla}$ which is the set of pure gauge connections. In the case $r = 1$, $N(n,1) = 1$, so one has only these two gauge orbits.

d) Formulae like (7) naturally appear in the double-bundle structures. See for example [9].

5. MODELS OF GAUGE THEORY.

5.1 <u>Classical euclidean Maxwell and Yang-Mills actions</u>. Throughout Section 5, $V = \mathbb{R}^{s+1}$ is the $(s+1)$-dimensional euclidean space-time with metric $ds^2 = \sum_{\mu=0}^{\mu=s} (dx^{\mu})^2$ and $\mathcal{A} = C^{\infty}(\mathbb{R}^{s+1})\otimes M_n(\mathbb{C})$. We recall here in the case $n=1$, i.e. $\mathcal{A} = C^{\infty}(\mathbb{R}^{s+1})$, the definition of the Maxwell action and, in general that of the $U(r)$-Yang-Mills action.

The Maxwell action is an action for connections on a $U(1)$-principal bundle over $\mathbb{R}^{s+1}$. One can also say that it is an action for hermitian connections on a hermitian vector bundle of rank one over $\mathbb{R}^{s+1}$. Finally, since $\mathbb{R}^{s+1}$ is contractible, it is an action for hermitian connections on the free hermitian $C^{\infty}(\mathbb{R}^{s+1})$-module of rank one. Let ∇ be such a connection with component $A = A_{\mu}dx^{\mu} \in \Omega^1(V) = \Omega_D^1(C^{\infty}(\mathbb{R}^{s+1}))$, (the Maxwell potential), and component $F = \frac{1}{2} F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}$ of the curvature ∇^2, (the corresponding electromagnetic field). One has $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. The Maxwell action $S(\nabla)$ for ∇ is

$$S(\nabla) = \|\nabla^2\|^2 = \frac{1}{4}\int \sum_{\mu,\nu} (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})^2 \, d^{s+1}x \tag{8}$$

This action is gauge invariant, positive, and vanishes only on the gauge orbit of pure gauge connections. Two connections in the same gauge orbit are considered as physically equivalent.

In the same way, the U(r)-Yang-Mills action is an action for hermitian connections on the free hermitian $C^{\infty}(\mathbb{R}^{s+1})$-module of rank r. If ∇ is such a connection with component $A = A_{\mu}dx^{\mu} \in \Omega^1(V)\otimes M_r(\mathbb{C})$, the Yang-Mills action is given by

$$S(\nabla) = -\frac{1}{4}\int \sum_{\mu,\nu} \frac{1}{r}\mathrm{Tr}\,((\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + [A_{\mu},A_{\nu}])^2) \, d^{s+1}x \tag{8'}$$

This action is again gauge invariant, positive, and vanishes only if ∇ is a pure gauge connection. It coincides with Maxwell action for $r=1$.

5.2 <u>Maxwell action for</u> $\mathcal{A} = C^{\infty}(\mathbb{R}^{s+1})\otimes M_n(\mathbb{C})$. It is natural to generalize the Maxwell action for arbitrary positive integer n as $\|\nabla^2\|^2$ on hermitian connection ∇ on the free hermitian $\mathcal{A}$-module of rank one. Let $\omega \in \Omega_D^1(\mathcal{A})$ be the component of ∇, then $\|\nabla^2\|^2$ means

$\langle d\omega + \omega^2 \mid d\omega + \omega^2 \rangle$ with the scalar product defined in 3.3 on $\Omega_D(\mathcal{A})$.

Since from 3.3, we know that there is an overall scale factor $\left(\frac{1}{m}\right)^{n^2-1}$ in front of this scalar product we define the generalized Maxwell action as $S(\nabla) = (m)^{n^2-1} \langle d\omega + \omega^2 \mid d\omega + \omega^2 \rangle$. Writing again ω as $\omega = A_\mu dx^\mu + (B_k - iE_k)\,\theta^k$ with antihermitian nxn-matrix-valued functions A_μ, $\mu \in \{0,1,\dots,s\}$ and B_k, $k \in \{1,\dots,n^2-1\}$, $S(\nabla)$ is given by

$$S(\nabla) = -\frac{1}{4n}\int \sum_{\mu,\nu} \mathrm{Tr}\,((\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu])^2) -$$

$$-\frac{m^2}{2n}\int \sum_{\lambda,k} \mathrm{Tr}((\partial_\lambda B_k + [A_\lambda, B_k])^2) - \frac{m^4}{4n}\int \sum_{k,\ell} \mathrm{Tr}(([B_k, B_\ell] - \sum_m C_{k\ell m} B_m)^2)$$

which can also be written

$$S(\nabla) = -\int_{\mathbb{R}^{s+1}} \frac{1}{4n}\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{2n}\mathrm{Tr}((\nabla_\lambda \phi_k)(\nabla^\lambda \phi^k))) +$$

$$+ \frac{1}{4n}\sum_{k,\ell} \mathrm{Tr}(([\phi_k, \phi_\ell] - m\sum_m C_{k\ell m}\phi_m)^2) \qquad (9)$$

where

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F^{\mu\nu}$, $\phi_k = m\, B_k$, $\nabla_\lambda \phi_k = \partial_\lambda \phi_k + [A_\lambda, \phi_k] = \nabla^\lambda \phi^k$.

Under a gauge transformation $\nabla \mapsto \nabla^U$, $U \in \mathcal{U}_1$, the A_μ transform as $A_\mu \mapsto U^{-1}A_\mu U + U^{-1}\partial_\mu U$, the ϕ_k transform as $\phi_k \mapsto U^{-1}\phi_k U$ and the $\nabla_\lambda \phi_k$ transform as $\nabla_\lambda \phi_k \mapsto U^{-1}(\nabla_\lambda \phi_k)U$. The action (9) is gauge invariant, positive and, for $n \geqslant 2$ vanishes, (in view of 4.5), on the gauge orbit of $(A_\mu = 0,\ \phi_k = 0)$ and on the gauge orbit of $(A_\mu = 0\ ,\ \phi_k = i m E_k)$.

5.3 <u>Discussion</u>. The action (9) can be interpreted as the euclidean action of a field theory on $\mathbb{R}^{s+1}$. It is then the euclidean action for a U(n)-Yang-Mills field minimaly coupled to n^2-1 scalar fields ϕ_k with values in the adjoint representation of U(n) which interact among themselves through a quartic polynomial potential.

We now assume that $n \geqslant 2$ and $s+1 \geqslant 2$. Then the two gauge orbits where the action vanishes are separated by an infinite barrier; there is no instanton interpolating between these two gauge orbits. This follows from the translation invariance. Therefore, by standard arguments [7] , each of these orbits corresponds to a vacuum for the corresponding quantum field theory in Minkowski space. Let Ω_0 be the vacuum corresponding to the gauge orbit of $(A_\mu = 0, \phi_k = 0)$ and let Ω_1 be the one corresponding to the gauge orbit of $(A_\mu = 0, \phi_k = i m E_k)$.

To specify a quantum theory, one has to choose a vacuum. Then in order to develop the theory, one has to use the field variables adapted to the corresponding vacuum sector. These field variables must vanish up to a gauge transformation on the gauge orbit corresponding to the chosen vacuum in order that the vacuum expectation values of the associated quantum fields vanish up to a gauge transformation.

Thus the variables A_μ and ϕ_k are adapted to the vacuum sector of Ω_0 corresponding to the gauge orbit of $(A_0 = 0,\ \phi_k = 0)$. In this sector, one has an ordinary massless U(n)-Yang-Mills field described by the A_μ

minimally coupled to the fields ϕ_k which are massive with the same mass $m_\phi = nm$.

The variables adapted to the vacuum sector of Ω_1 are the A_μ and the $\Psi_k = \phi_k - im\, E_k$. The translation $\phi_k \mapsto \Psi_k$ gives a quadratic term in the traceless part (i.e. the SU(n) part) of the A_μ which becomes massive with the mass $m_A = \sqrt{2}\, nm$. The U(1) part of the A_μ remains massless and the mass spectrum of the Ψ_k becomes complicated. We shall describe this spectrum in the case n=2.

5.4 The case n=2 in the sector of Ω_1. The vacuum Ω_1 corresponds to the gauge orbit of pure gauge connections on the free hermitian $\mathcal{A}$-module of rank one. The vacuum sector of Ω_1 is therefore very natural from the point of view of the underlying non-commutative differential geometry. We now assume that $\mathcal{A} = C^\infty(\mathbb{R}^{s+1}) \otimes M_2(\mathbb{C})$ and we compute the mass spectrum of the Ψ_k (for Ω_1). For that we write Ψ_k as $\Psi_k = i(\Psi_k^0 \mathbb{1} + \Psi_k^\ell E_\ell)$ and decompose Ψ_k^ℓ into its irreductible parts as $\Psi_k^\ell = \tau\ \delta_k^\ell + \sigma_k^\ell + \alpha_k^\ell$ where $\tau = \frac{1}{3}\Psi_\ell^\ell$, (σ_k^ℓ) is symmetric and traceless and (α_k^ℓ) is antisymmetric. One then obtains from (9) and $\Psi_k = \phi_k - im\, E_k$ the following mass spectrum. The fields Ψ_k^0 have mass $m_0 = 2m$, the field τ has mass $m_\tau = 2m$, the fields σ_k^ℓ have mass

$m_\sigma = 4m$ and the fields α_k^ℓ are massless $m_\alpha = 0$. Notice that, in contrast to the ϕ_k, the Ψ_k transform inhomogeneously under a gauge transformation and that one can fix the gauge by imposing $\alpha_k^\ell = 0$.

5.5 Generalization. One can generalize similarly the U(r)-Yang-Mills action by writing the action for a hermitian connection on the free hermitian $\mathcal{A}$-module of rank r. The action has again the form (9) but now the A_μ and the ϕ_k are $nr \times nr$-antihermitian-matrix-valued. Thus, using the theorem 5.4, there are as many gauge orbits of connections on which the action vanishes as there are unitary classes of anti-hermitian representations of SU(n) in $\mathbb{C}^{nr}$. One thus has vacua Ω_α, $\alpha \in \{0,1,\dots,N(n,r)\}$, for the quantum theory. The number $N(n,r)$ grows very quickly with r for $n \geqslant 2$.

6. CONCLUSION

For $A = C^\infty(\mathbb{R}^4) \otimes M_n(\mathbb{C})$, i.e. on 4-dimensional space-time with $n=2$, the theory described in 5.2, 5.3, 5.4 has similarities with the bosonic part of the standard model of electroweak theory. The ϕ_k plays the role of the Higgs fields and the sector of Ω_1 is similar to the broken phase. One has then a U(1)xSU(2) gauge theory and the mechanism which produces a mass for the SU(2) part of the gauge field is very similar to the Higgs mechanism. There are however two main differences. The first one is that here one has two stable gauge invariant vacua. The second one is that since the ϕ_k or the Ψ_k are component of a hermitian connection, they are antihermitian and thus, they do not interact with the electromagnetic field i.e. with the U(1) part of the A_μ . Thus there is nothing her like the Weinberg angle and the U(1)-gauge field is completely decoupled.

From the point of view of perturbation theory in $\mathbb{R}^4$, the theory we have presented is renormalizable. To carry out the renormalization program one has to use standard B.R.S. technique. However the usual B.R.S. invariance does not forbid terms like $\mathrm{Tr}\,(\phi_k^2)$ with arbitrary coefficients. These would break the form of (9) which is the square norm of a curvature and one must therefore find an extended B.R.S. or some other invariance which takes into account the fact that the action is a functional of a curvature. Another point which we did not discuss here is the theory of spinor fields in the context of our model. Work on these points is currently is progress.

In [10] we give an informal discussion of the models of gauge theory presented here with a description of the analogue of the scalar field for $\mathcal{A} = C^\infty(\mathbb{R}^{s+1}) \otimes M_n(\mathbb{C})$ and we discute the analogies and the differences of our work with the theories of Kaluza-Klein type.

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