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Classical Bosons in a Noncommutative Geometry

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Abstract: The classical theory of the scalar field is developed within the context of a noncommutative geometry. A noncommutative extension of abelian gauge theory is proposed and is compared with standard nonabelian gauge theory.

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1. Introduction The basic idea of noncommutative geometry is to replace an algebra of functions defined on a manifold by an abstract associative algebra $\mathcal{A}$ which is not necessarily commutative. We refer to [1] for a general introduction to this subject and for references to the previous literature. One can then attempt to reformulate the objects of interest in differential geometry such as vector fields, differential forms, metrics and connections, in terms of operations involving only $\mathcal{A}$. We refer to [2] for a simple example of this and for references to the previous literature. Classical field theory, especially gauge-field theory, can be very well written in a geometric language. For example, a scalar field is a function, a gauge potential is a connection and the gravitational potential is a metric. Therefore an obvious thing to try to do is to develop a noncommutative version of classical field theory simply by replacing the objects in the commutative case which are used to describe the theory by the corresponding noncommutative objects. For example a scalar field becomes an element of an abstract algebra $\mathcal{A}$ which is no longer an algebra of functions. The scalar field then is no longer a function on space-time.

Any supersymmetric theory is based on a noncommutative algebra, of a very special sort however, which is more properly described as graded commutative. The geometry which we use has a supersymmetric extension. This will be developed in a subsequent article.

If $\mathcal{A}$ is commutative then under certain very general conditions, it can be shown to be an algebra of functions. If $\mathcal{A}$ is too abstract or too different from the ordinary algebra of functions on space-time, it is impossible to interpret its elements in terms of classical observables. As a compromise between these two extremes, and for other reasons to be given below, we propose here to develop classical field theory using as algebra the tensor product of the algebra M_n of $n \times n$ complex matrices and an algebra $\mathcal{C}$ of functions on space-time:

$$\mathcal{A} = M_n \otimes \mathcal{C}.$$

In two previous articles [2, 3] the basic noncommutative geometry was developed using this particular algebra and within the context of this geometry a theory of gauge bosons was formulated. We propose here to present the contents of these articles in a less formal fashion and to extend their application to the scalar field. We also present the results from the point of view of Kaluza-Klein theory.

In Section 2 we recall the basic definitions and some of the results of the previous articles. The field which presents the greatest departure from the commutative case is the gauge field. The most interesting result here is the existence of a multiple vacuum structure which allows for the possibility of two phases, a symmetric phase with massless gauge bosons and a broken phase with some massive ones. In contrast to usual theories with spontaneous symmetry breaking, both of the phases which we find here are stable and separated by a potential barrier. There is an analog of the Higgs field and a Higgs potential which is contained as a term in the generalized curvature 2-form. This is described in detail in Section 3. In Section 4 we define the basic objects of interest in classical field theory, the field itself, the action and the equations of motion, in the case of a scalar field. We find that because of our choice of algebra the scalar field is in fact a multiplet of fields. If the lagrangian contains no explicit mass term the multiplet consists of one massless field and a certain number of

massive ones. This structure had already been encountered in the case of the gauge fields. In Section 5 we give a Kaluza-Klein unification of the gauge bosons with the gravitational field. In Section 6 we briefly compare the noncommutative gauge models for $n = 2, 3$ with the standard nonabelian gauge models of the electroweak and strong interactions. The noncommutative version of a fermion field is not as straightforward as that of a boson and we have left it to a subsequent article.

2. Motivation and Mathematical Preliminaries In the conventional formulation of quantum field theory as the theory of formally quantized classical fields on a classical Minkowski space-time, ultraviolet divergences arise when one attempts to measure the amplitude of field oscillations at a precise given point. This was one of the original motivations of the authors for studying noncommutative geometry. Suppose that one could modify the microscopic structure of space with the hypothesis that at a sufficiently small length scale κ the coordinates which define the position of a point become non-commuting operators. This means that, in particular, it would be impossible to measure exactly the position of a particle since the three space coordinates could not be simultaneously diagonalized. Then it is conceivable that the length scale κ could, as in the case of a lattice structure, be used to introduce an effective ultraviolet cut-off. The extra structure which a point acquires at length scales less than κ could be considered as the origin of the spin of a particle. This possibility has been explored [4]. See also [5]. It could also be identified with the origin of isospin. It is this possibility which we pursue here. A more ambitious effort would be to introduce a richer structure, using a more complicated algebra, which could unify spin with isospin. This richer structure would certainly be necessary were one to hope to introduce an ultraviolet cut-off.

In the introduction we defined the algebra $\mathcal{A}$. We shall choose $\mathcal{C}$ to be the algebra of smooth complex-valued functions on $\mathbf{R}^4$. The extra factor M_n is the origin of the extra structure which a point acquires. The elements of the algebra $\mathcal{A}$ are what replace the functions on $\mathbf{R}^4$. In particular the nearest object which we have to a coordinate is an element of this algebra. This means that the position of a particle, for example, no longer has a well-defined meaning. Since we certainly wish this to be so at macroscopic scales, we must require that the scale κ be not much greater than a typical Compton wave length. In other words, the fuzziness which the internal structure gives a point in space-time cannot be much greater than the quantum uncertainty in the position of a particle. The Lorentz group acts on $\mathcal{A}$. This means in particular that directions are well-defined. Our space-time looks like a crystal then which has a homogeneous distribution of dislocations but no disclinations. We can pursue this solid-state analogy and think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by course-graining over scales less than κ. They break down and must be replaced by elements of the algebra $\mathcal{A}$ when one considers phenomena on these scales.

The Lie algebra $D(\mathcal{C})$ of vector fields on the manifold $\mathbf{R}^4$ can be identified with the algebra of derivations of $\mathcal{C}$, that is, with the algebra of linear maps of $\mathcal{C}$ into itself which satisfy the Leibnitz rule. This algebra is the most important mathematical object which one uses when one studies classical fields on $\mathbf{R}^4$ and their dynamics. To study these fields then in the noncommutative case we must consider the derivations $D(\mathcal{A})$ of the algebra $\mathcal{A}$. This Lie algebra was shown in [3] to be the direct sum of the

ordinary derivations of C and the C-module generated by the inner derivations of M_n. A problem which arises which was not present in the commutative case is the fact that the derivations no longer form a module over the algebra $\mathcal{A}$. A derivation when multiplied by an element of $\mathcal{A}$ is no longer a derivation. This is in sharp contrast to the normal situation where a vector field multiplied by a function remains a vector field. We avoid this problem as much as possible by working with the generalizations of differential forms, which are dual to the derivations in a sense which we shall give below.

We recall the notation. Let λ_a, for $1 \leq a \leq n^2 - 1$, be a basis of the Lie algebra of the special unitary group in n dimensions, chosen so that the structure constants $C^a{}_{bc}$ are real. The Killing metric is given by $g_{ab} = -Tr(\lambda_a \lambda_b)$. We shall raise and lower indices with this metric. The set λ_a is a set of generators of the matrix algebra M_n. It is not a minimal set but it is convenient because of the fact that the derivations $e_a = \kappa^{-1} ad(\lambda_a)$ form a basis over the complex numbers for the derivations of M_n. They satisfy the commutation relations

$$[e_a, e_b] = m\, C^c{}_{ab}\, e_c. \tag{2.1}$$

The mass scale m is defined to be the inverse of the length scale κ. As a particular case consider M_{2^k} with for example $k = 2$. It can be identified with the ordinary Clifford algebra and as such it has also the 4 euclidean Dirac matrices as generators. We mention this as an example of a natural set of generators fewer in number than the set given above. For any n, M_n can always be generated in fact by only 3 elements, but for $n > 2$ these generators have no invariance properties.

Let x^μ be coordinates of $\mathbf{R}^4$. Then the set (x^μ, λ^a) is a set of generators of the algebra $\mathcal{A}$. We define the exterior derivative of an element of $\mathcal{A}$ as usual. For example, if f is an element of M_n, then df is defined [6] by the formula

$$df(e_a) = e_a(f). \tag{2.2}$$

This means in particular that

$$d\lambda^a(e_b) = m\,[\lambda_b, \lambda^a] = m\, C_{cb}{}^a \lambda^c. \tag{2.3}$$

We define the set of 1-forms $\Omega^1(\mathcal{A})$ to be the set of all elements of the form $f dg$ or $dg f$, with f and g in $\mathcal{A}$ subject to the relations $d(fg) = (df)g + f dg$. We similarly define $\Omega^1(M_n)$ and $\Omega^1(C)$, the definition of the latter being of course the usual one. The n-forms $\Omega^n(\mathcal{A})$ and the generalization $\Omega(\mathcal{A})$ of the algebra of differential forms are defined in [6]. See also [2, 3]. The space $\Omega(\mathcal{A})$ is written $\Omega_D(\mathcal{A})$ in the previous articles since $\Omega(\mathcal{A})$ has another meaning in the mathematical literature. Other minor differences with these articles follow from the different normalization chosen for the generators of M_n

The algebra $\mathcal{A}$ can be viewed as an algebra of matrix-valued functions and it is convenient to consider it as such at times. However then one would not use the operator d which has just been defined. The kernel of d, the elements f with $df = 0$, is the set of f which are not only constant when considered as matrix-valued functions on $\mathbf{R}^4$, but which are also multiples of the identity.

The set of $d\lambda^a$ forms a system of generators of $\Omega^1(M_n)$ as a left or right module, but it is not a convenient one. For example $\lambda^a d\lambda^b \neq d\lambda^b \lambda^a$. However because of the particular structure of M_n [2] there is another system of generators completely characterized by the equation

$$\theta^a(e_b) = \delta^a_b. \tag{2.4}$$

It is related to the $d\lambda^a$ by the equations

$$d\lambda^a = m\, C^a{}_{bc}\, \lambda^b \theta^c, \qquad \theta^a = \kappa\, \lambda_b \lambda^a d\lambda^b, \tag{2.5}$$

and it satisfies the same structure equations as the components of the Maurer-Cartan form on the special unitary group SU_n:

$$d\theta^a = -\frac{1}{2} m\, C^a{}_{bc}\, \theta^b \theta^c. \tag{2.6}$$

The product on the right-hand side of this formula is the product in $\Omega(M_n)$. Although this product is not in general antisymmetric, because of the relation (2.4) we have

$$\theta^b \theta^a = -\theta^a \theta^b.$$

Also the θ^a commute with the elements of M_n and $\Omega^1(M_n)$ can be identified with the tensor product of M_n and the dual of the vector space of derivations. The subalgebra $\Omega^*(M_n)$ of $\Omega(M_n)$ generated by the θ^a is an exterior algebra. Formula (2.6) means that it is a differential subalgebra. Since we shall only use $\Omega^*(M_n)$ in what follows we shall write the product as a wedge product:

$$\theta^a \theta^b = \theta^a \wedge \theta^b.$$

Choose a basis $\theta^\alpha_\lambda dx^\lambda$ of $\Omega^1(\mathcal{C})$ over $\mathcal{C}$ and let e_α be the pfaffian derivations dual to θ^α. Set $i = (\alpha, a)$, $1 \leq i \leq 4 + n^2 - 1$, and introduce $\theta^i = (\theta^\alpha, \theta^a)$ as generators of $\Omega^1(\mathcal{A})$ as a left or right $\mathcal{A}$ - module and $e_i = (e_\alpha, e_a)$ as a basis of $D(\mathcal{A})$ over $\mathcal{C}$. We can write $\Omega^1(\mathcal{A})$ as a direct sum

$$\Omega^1(\mathcal{A}) = \Omega^1_V \oplus \Omega^1_H, \tag{2.7}$$

where we define

$$\Omega^1_V = \mathcal{A} \otimes_{M_n} \Omega^1(M_n) = \mathcal{C} \otimes \Omega^1(M_n), \qquad \Omega^1_H = \mathcal{A} \otimes_{\mathcal{C}} \Omega^1(\mathcal{C}) = M_n \otimes \Omega^1(\mathcal{C}).$$

The vertical part Ω^1_V has basis θ^a and the horizontal part Ω^1_H has basis θ^α. We shall decompose the exterior derivative into vertical and horizontal parts also:

$$d = d_V + d_H.$$

We shall introduce the quadratic form, of signature $d - 2$, given by

$$ds^2 = \eta_{ij} \theta^i \otimes \theta^j = \eta_{\alpha\beta} \theta^\alpha \otimes \theta^\beta + g_{ab} \theta^a \otimes \theta^b. \tag{2.8}$$

The $\eta_{\alpha\beta}$ is the Minkowski metric. We shall refer to this quadratic form as a metric although it contains two terms of a slightly different nature. To within a rescaling the g_{ab} are the components of the unique metric g_V for M_n with respect to which all the derivations e_a are Killing derivations. Let $X = X^a e_a$ be an arbitrary derivation. It is easy to see that

$$L_X g_V = 0$$

and that conversely any metric for M_n which satisfies this equation must be a constant multiple of g_V.

From the basis elements θ^a we can construct a 1-form θ in Ω^1_V,

$$\theta = -m\lambda_a \theta^a, \tag{2.9}$$

which from (2.5) and (2.6) satisfies the zero-curvature condition

$$d\theta + \theta^2 = 0. \tag{2.10}$$

We shall see below in Section 3 that θ is gauge invariant. It satisfies with respect to the algebraic exterior derivative d_V similar conditions to those which the Maurer-Cartan form satisfies with respect to ordinary exterior derivation on the group SU_n. We have a map of the trace-free elements of M_n onto the derivations of M_n given by $f \mapsto X = ad(f)$. The 1-form θ can be defined, without any reference to the θ^a, as the inverse map: $\theta(X) = -f$. Using it, the vertical component of the metric can be given by the equation

$$g_V(X,Y) = -\kappa^2\, tr(\theta(X)\theta(Y)),$$

where X, Y are derivations of M_n. Set $X = ad(f)$. Then using (2.10) we find

$$L_X\theta = i(X)d\theta + di(X)\theta = -(df + [\theta, f]) = 0.$$

So the 1-form θ is invariant under all the derivations of M_n.

The generators θ^a of $\Omega^1(M_n)$ can be considered as a sort of moving frame. If we compare equation (2.6) with the first structure equations for this frame,

$$d\theta^a + \omega^a{}_b \wedge \theta^b = \Theta^a,$$

we see that if we require the torsion form Θ^a to vanish then the internal structure is like a curved space with a linear connection given by

$$\omega^a{}_b = -\frac{1}{2} m\, C^a{}_{bc}\, \theta^c. \tag{2.11}$$

We have then a covariant derivative D_a which differs from e_a. Alternatively we could require that the connection $\omega^a{}_b$ vanish. We would have then a torsion form given by

$$\Theta^a = -\frac{1}{2} m\, C^a{}_{bc}\, \theta^b \wedge \theta^c. \tag{2.12}$$

We have in this case a covariant derivative D_a which is equal to e_a in the absence of gauge couplings but we have

$$[D_a, D_b] = m\, C^c{}_{ab}\, D_c. \tag{2.13}$$

In the rest of the article we shall choose the solution (2.11), to avoid having an extra torsion form. If α is a 1-form $\alpha = \alpha_a \theta^a$ then we have by definition, in the absence of torsion,

$$d\alpha = D_a \alpha_b \, \theta^a \wedge \theta^b \tag{2.14}$$

with

$$D_a \alpha_b = e_a \alpha_b - \frac{1}{2} m \, C^c{}_{ab} \, \alpha_c. \tag{2.15}$$

The equations we have given above are with respect to an arbitrary basis λ^a but they are all tensorial in character with respect to a change of basis

$$\lambda^a \mapsto \lambda'^a = A^a_b \lambda^b, \qquad (A^a_b) \in GL_{n^2-1}. \tag{2.16}$$

In particular the covariant derivative (2.15) transforms as it should. The unusual fact here is that the connection transforms also as a tensor and each of the terms on the right-hand side of (2.15) is tensorial in character separately. This is related to the fact that on the factor M_n of our algebra there is no notion of a point and no analog of local variation. Each θ^a corresponds to an arbitrary globally defined moving frame in the commutative case and the transformations of the moving frame θ^a correspond to the set of all local transformations. We speak of a linear connection even though the corresponding moving frames do not vary. The change of basis (2.16) is the equivalent in M_n of a coordinate transformation in $\mathcal{C}$. We can suppose that a basis has been chosen so that the Killing metric g_{ab} is equal to the euclidean metric δ_{ab}.

One can also consider the automorphisms of M_n, given by

$$\lambda^a \mapsto \lambda'^a = g^{-1} \lambda^a g, \qquad g \in GL_n. \tag{2.17}$$

To simplify we mention only the infinitesimal transformations

$$\lambda^a \mapsto \lambda'^a = \lambda^a - [f, \lambda^a] \qquad g \simeq 1 + f.$$

We see that

$$\theta'^a = \theta^a - L_X \theta^a, \tag{2.18}$$

and in general, for any n-form α

$$\alpha' = \alpha - L_X \alpha.$$

We noticed above that the 1-form θ is invariant under the automorphisms of M_n.

We introduced above the generators θ^i of $\Omega^1(\mathcal{A})$ which consist of θ^a given by (2.5) and a basis θ^α of $\Omega^1(\mathcal{C})$. We mentioned the invariance with respect to a change of basis in M_n which gives rise to a linear transformation of the θ^a. We can also change θ^α by a local Lorentz transformation. There is a much larger group of transformations, the local rotations of the entire set θ^i, which mixes the ordinary moving frame θ^α with the algebraic generators θ^a of $\Omega^1(M_n)$:

$$\theta'^i = \Lambda^i_j(x^\lambda) \theta^j. \tag{2.19}$$

The lagrangians we shall consider are invariant under this group. We shall not use this fact however since we shall not introduce curvature in the space-time metric. We mention it only to motivate the BRS operator which we define in the next section.

3. Gauge Fields In the commutative case a connection ω on the trivial principal U_1-bundle equipped with the associated canonical flat connection is an anti-hermitian 1-form which can be split as the sum of a vertical part, the Maurer-Cartan form $d\alpha$ on U_1, and a horizontal part, a 1-form on the base manifold,

$$\omega = d\alpha + A. \tag{3.1}$$

The gauge potential A is an element of $\Omega^1(\mathcal{C})$ and using it we can construct a covariant derivative on an associated vector bundle. The notion of a vector bundle can be generalized to the noncommutative case as an $\mathcal{A}$-module which in its simplest form, a free module of rank 1, can be identified with $\mathcal{A}$ itself. This is in fact the natural generalization to the algebra we are considering of a trivial U_1-bundle since M_n has replaced $\mathbf{C}$ in our models. So the U_n gauge symmetry we shall use below comes not from the rank of the vector bundle, which we shall always choose to be equal to 1, but rather from the factor M_n in our algebra $\mathcal{A}$. The noncommutative generalization of A is an anti-hermitian element of $\Omega^1(\mathcal{A})$, which we saw in the previous section in turn can be split as the sum of two parts, called also vertical and horizontal. We shall here designate by ω such an element of $\Omega^1(\mathcal{A})$ since we wish to reserve the letter A and the name gauge potential for the horizontal part in this latter sense. We write then

$$\omega = \theta + \phi + A, \tag{3.2}$$

where ϕ is an element of Ω^1_V and A is an element of Ω^1_H. The field ϕ is the Higgs field. We have noted that θ is in many respects like a Maurer-Cartan form. Formula (3.2) with $\phi = 0$ and formula (3.1) are formally similar but the meaning of the words horizontal and vertical in the two cases is not the same. We have then a bundle over a space which itself resembles a bundle. This double-bundle structure, which is what gives rise to a quartic Higgs potential as we shall see below, has been investigated in previous articles [7, 8].

Let $\mathcal{G}$ be the group of invertible elements of $\mathcal{A}$, considered as functions on $\mathbf{R}^4$ with values in GL_n and g an element of $\mathcal{G}$. We can consider the algebra $\mathcal{A}$ as a left $\mathcal{G}$-module, a right $\mathcal{G}$-module or as a bimodule with respectively the gauge transformations

$$f \mapsto g^{-1}f, \qquad f \mapsto fg, \qquad f \mapsto g^{-1}fg.$$

This means that f is respectively an isospinor, a co-isospinor or an isovector. If $\mathcal{G}$ acts on $\mathcal{A}$ from the left or right it takes $\mathcal{C} \otimes 1$ onto a dense subset of $\mathcal{A}$. Let $\mathcal{U}_n$ be the subgroup of $\mathcal{G}$ of elements which satisfy $gg^* = 1$. We shall choose it to be the group of local gauge transformations.

A gauge transformation defines a mapping of $\Omega^1(\mathcal{A})$ into itself of the form

$$\omega' = g^{-1}\omega g + g^{-1}dg. \tag{3.3}$$

We define

$$\begin{aligned} \theta' &= g^{-1}\theta g + g^{-1}d_V g, \\ A' &= g^{-1}Ag + g^{-1}d_H g, \end{aligned} \tag{3.4}$$

and so ϕ transforms under the adjoint action of $\mathcal{U}_n$:

$$\phi' = g^{-1}\phi g.$$

It can be readily seen that in fact θ is invariant under $\mathcal{G}$,

$$\theta' = \theta, \tag{3.5}$$

and so the transformed potential ω' is again of the form (3.2).

The generalization of the globally defined maps of a manifold onto itself in the commutative case is the set of automorphisms of the algebra $\mathcal{A}$. This consists of globally defined transformations of $\mathcal{C}$ onto itself and of the mappings of M_n onto itself of the form $ad(g^{-1})$, for g in $\mathcal{G}$. We see then that gauge transformations can be identified with a subset of generalized coordinate transformations. This is a standard Kaluza-Klein result. Care must be taken to distinguish between λ^a as generator of the algebra $\mathcal{A}$ and λ_a in the expression for the basis e_a of the derivations of $\mathcal{A}$. Both of these transform under automorphisms of $\mathcal{A}$ but only the former changes under a local gauge transformation. The relation between λ^a and e_a is similar to that which exists between the coordinate x^μ and the partial derivative ∂_μ. The difference lies in the fact that e_a is a linear expression in λ_a. When e_a acts on an element f of $\mathcal{A}$ it yields the components of a 1-form. In the expression (2.9) for θ, λ_a transforms as the derivative of a scalar but because of (2.10) there is no f such that $\theta = df$.

The fact that θ is invariant under a gauge transformation means in particular that it cannot be made to vanish by a choice of gauge. We have then a potential with vanishing curvature but which is not gauge-equivalent to zero. If M_n were an algebra of functions over a compact manifold, the existence of such a 1-form would be due to the non-trivial topology of the manifold.

We define the curvature 2-form Ω and the field strength F as usual

$$\Omega = d\omega + \omega^2, \quad F = d_H A + A^2.$$

Because of the identity

$$d_V A + \theta A + A\theta = 0,$$

which follows from (2.5), we have the following relation between Ω and F:

$$\Omega = F + D_H\phi + (D_V\phi - \phi^2), \tag{3.6}$$

where we have defined the covariant exterior derivative

$$D\phi = d\phi + \omega\phi + \phi\omega.$$

and decomposed it into horizontal and vertical parts. In terms of components, with $\phi = \phi_a\theta^a$ and $A = A_\alpha\theta^\alpha$ and with

$$\Omega = \frac{1}{2}\Omega_{ij}\theta^i \wedge \theta^j, \quad F = \frac{1}{2}F_{\alpha\beta}\theta^\alpha \wedge \theta^\beta, \tag{3.7}$$

we find

$$\Omega_{\alpha\beta} = F_{\alpha\beta}, \quad \Omega_{\alpha a} = D_\alpha\phi_a, \quad \Omega_{ab} = D_{[a}\phi_{b]} - [\phi_a, \phi_b]. \tag{3.8}$$

The covariant derivative here is the sum of the gauge term and the extra contribution (2.15):

$$D_a\phi_b = e_a\phi_b + [\omega_a, \phi_b] - \frac{1}{2}m\, C^c{}_{ab}\,\phi_c. \tag{3.9}$$

This can be rewritten

$$D_a\phi_b = [\phi_a, \phi_b] - \frac{1}{2}m\, C^c{}_{ab}\,\phi_c$$

and it vanishes for the value

$$\phi_a = \frac{1}{2}m\lambda_a. \tag{3.10}$$

This fact suggests yet another natural linear connection. If we require that

$$\nabla_a\phi = (\nabla_a\phi_b)\theta^b = 0$$

then we obtain a linear connection with torsion which depends on the commutator of the Higgs field. The torsion depends also on the Higgs field and vanishes only at the value given by (3.10). Using (3.9) we can rewrite the expression for Ω_{ab}:

$$\Omega_{ab} = [\phi_a, \phi_b] - m\, C^c{}_{ab}\,\phi_c. \tag{3.11}$$

The Bianchi identities follow as in the usual commutative geometry. They depend only on the identity $d^2 = 0$.

The gauge-invariant action for the gauge potentials A and the Higgs fields ϕ is given by

$$S = \frac{1}{4}Tr\int(\Omega_{ij}\Omega^{ij}). \tag{3.12}$$

The integration is over space-time and the trace is what replaces this integration on the factor M_n of $\mathcal{A}$. This is the simplest gauge action which can be written down in the geometries which we consider. Written out explicitly using (3.8), the lagrangian becomes

$$\mathcal{L} = \frac{1}{4}Tr(F_{\alpha\beta}F^{\alpha\beta}) + \frac{1}{2}Tr(D_\alpha\phi_a D^\alpha\phi^a) - V(\phi), \tag{3.13}$$

where the Higgs potential $V(\phi)$ is given by

$$V(\phi) = -\frac{1}{4}Tr(\Omega_{ab}\Omega^{ab}). \tag{3.14}$$

It is a quartic polynomial in ϕ which is fixed and has no free parameters apart from the mass scale m.

Consider an infinitesimal gauge transformation $g \simeq 1 + \chi$. If we use the identity

$$D_a\chi = [\phi_a, \chi] \tag{3.15}$$

we have

$$\delta A_\alpha = D_\alpha\chi, \qquad \delta\theta = 0, \qquad \delta\phi_a = [\phi_a, \chi]. \tag{3.16}$$

The lagrangian (3.13) remains invariant under δ, but so does the term

$$\mathcal{L}' = Tr(\phi_a\phi^a) \tag{3.17}$$

and similar terms of higher order in the Higgs field. To assure that such terms cannot appear as possible counterterms when the theory is quantized we shall include the transformations (2.19) and impose invariance under the following BRS operator:

$$\delta\omega_i = D_i c + \omega_j c_i^j. \tag{3.18}$$

This must be applied to the general lagrangian in (3.12) since the form (3.13) is obtained using the identity (3.15) which is not invariant under (2.19). The extra ghosts are components of a multiplet c_j^i which one must introduce to incorporate the invariance (2.19). Since we have not introduced the gravitational field we can choose the c_j^i to be constants. Under the transformation (3.18) we find that

$$\delta\mathcal{L}' \neq 0$$

and that the unique BRS-invariant lagrangian is given by (3.13).

From (3.11) we see that $V(\phi)$ vanishes for the values

$$\phi_a = 0, \qquad \phi_a = m\,\lambda_a. \tag{3.19}$$

Referring back to the expansion (3.2) of the connection we see that the first value corresponds to $\omega_V = \theta$ and the second value to $\omega_V = 0$. The orbit of the second value under the action of the gauge group can be identified with SU_n. The Higgs potential has therefore 2 absolute minima, a point at the origin and a submanifold of dimension $n^2 - 1$, which are separated by a potential barrior. There are therefore 2 stable phases.

We shall consider first the symmetric phase, $\phi_a = 0$ and then the broken phase, $\phi_a = m\,\lambda_a$. We expand the Higgs field and the gauge potential

$$\phi_a = \lambda_b \phi_a^b + \frac{i}{\sqrt{n}}\phi_a^0, \qquad A_\alpha = \lambda_a A_\alpha^a + \frac{i}{\sqrt{n}} A_\alpha^0$$

so that we can identify the basic modes, the elementary scalars (ϕ_a^b, ϕ_a^0) and the elementary vector bosons (A_α^a, A_α^0). In the symmetric phase the masses of all the Higgs modes are equal and they are real since the corresponding value of the potential is a stable minimum. Using the expression given above for Ω_{ab}, we find that the mass is given by

$$m_H^2 = nm^2. \tag{3.20}$$

The gauge bosons of course have vanishing mass in the symmetric phase.

In the broken phase, to calculate the Higgs masses we use the vertical part $\omega_a\theta^a$ of the connection as a new Higgs field and we expand it in terms of the basic field modes:

$$\omega_a = \lambda_b \omega_a^b + \frac{i}{\sqrt{n}}\omega_a^0.$$

The analysis here is rather messy and we give it only in the case $n = 2$. We decompose $\omega_{ab} = g_{bc}\omega_a^c$ into its irreducible parts, a trace, a trace-free symmetric and an antisymmetric part:

$$\omega_{ab} = \frac{\tau}{\sqrt{3}} g_{ab} + \sigma_{ab} + \alpha_{ab}.$$

The masses are respectively given by

$$m_\tau^2 = 2m^2, \quad m_\sigma^2 = 8m^2, \quad m_\alpha^2 = 0, \quad m_0^2 = 2m^2, \tag{3.21}$$

where m_0 is the mass of the modes ω_a^0. The 3 degrees of freedom in the α are the 3 gauge degrees of freedom. They correspond to the modes which have been absorbed as the longitudinal part of the now massive gauge bosons. In the broken phase the mass of the A^0 remains equal to zero. The mass term for the remaining vector bosons is

$$\mathcal{L}_{m_H} = \frac{1}{2} m^2 Tr([A_\alpha, \lambda_a][A^\alpha, \lambda^a]). \tag{3.22}$$

The mass of these bosons is given therefore by the equation

$$m_A^2 = 2nm^2. \tag{3.23}$$

If we compare this with (3.21) in the case $n = 2$, we obtain a numerical prediction. Some of the masses of the Higgs bosons are larger than that of the vector bosons by a factor $\sqrt{2}$ and some are smaller by the same factor.

We have two classical vacua separated by a potential barrier. There is however no instanton solution which tunnels between them since there is not even a smooth field configuration with finite action which could take as values $\omega = 0$ at $t \to -i\infty$ and $\omega = \theta$ at $t \to +i\infty$.

The potential $V(\phi)$ can be identified with the (euclidean) action of the vertical part ω of the connection. The trace replaces in the algebra M_n the integration over the points of the manifold in the usual case. If we define an instanton to be a finite-action field configuration, solution to the Yang-Mills field equations, then we have found all instantons. In the gauge $g = 0$ (See (3.7)) they are given by

$$\phi = 0, \qquad \phi = -\frac{1}{2}\theta, \qquad \phi = -\theta. \tag{3.24}$$

The first and last solutions have zero curvature and zero action. The second has positive action but is unstable.

4. The Scalar Field In ordinary classical field theory, a charged scalar field is a complex-valued function on space-time. It is an element of $\mathcal{C}$. Therefore we define here a scalar field f to be an element of $\mathcal{A}$. The gauge group $\mathcal{U}_n$ can be supposed to act from the left on f. Because f can be written as

$$f = \lambda_a f^a + \frac{i}{\sqrt{n}} f^0, \tag{4.1}$$

where f^a and f^0 are n^2 complex-valued functions, f is in fact a multiplet of n^2 ordinary scalar fields.

The appropriate action is the integral over space-time of the lagrangian given by

$$\mathcal{L} = \frac{1}{2} Tr(e_i f^* e^i f). \tag{4.2}$$

The equations of motion are

$$\Delta f = 0, \tag{4.3}$$

where we have introduced the generalized d'Alembertian or Laplace operator

$$\Delta = -e_i e^i. \tag{4.4}$$

This is the Laplace operator which acts on scalars. It has a straightforward generalization [2, 3] to an operator which acts on differential forms, the elements of $\Omega^*(\mathcal{A})$. We should have used the covariant derivative with respect to (2.11). The scalar field however depends only on the metric and not on a particular associated linear connection.

We split Δ as the sum of an internal or vertical part $\Delta^{(i)}$ and an external or horizontal part $\Delta^{(e)}$:

$$\Delta = \Delta^{(i)} + \Delta^{(e)}.$$

The complete spectrum of $\Delta^{(i)}$ can be readily calculated. It is given by

$$\begin{aligned} \Delta^{(i)} 1 &= 0, \\ \Delta^{(i)} \lambda_a &= 2nm^2 \lambda_a. \end{aligned} \tag{4.5}$$

So in the broken phase the scalar mode f^0 is massless and the other modes f^a have mass given by the formula

$$m_f^2 = 2nm^2. \tag{4.6}$$

We found already this mass-multiplet structure in the section devoted to the gauge bosons.

We introduce a coupling to the gauge potential through the normal minimal-coupling procedure,

$$e_i \rightarrow D_i = e_i + \omega_i.$$

With this coupling, the lagrangian (4.2) becomes gauge invariant. Because of the rigidity of the theory it is necessary to consider the minimally-coupled lagrangian. For example in the symmetric phase, in the absence of all external fields, the covariant derivative D_i becomes

$$D_a = e_a - m\lambda_a, \qquad D_\alpha = e_\alpha. \tag{4.7}$$

In this phase all the scalar-mode masses are equal and they are given by

$$m_f^2 = \frac{1}{n}(n^2 - 1)m^2. \tag{4.8}$$

The internal laplacian $\Delta^{(i)}$ is proportional to the identity matrix in M_n.

In the broken phase the masses remain as given by (4.6) since ω_i vanishes in the absence of external fields.

We have assumed that the scalar field transforms under the fundamental representation of the gauge group, that it is an isospinor:

$$f \rightarrow g^{-1} f. \tag{4.9}$$

This representation splits into n irreducible representations which become mixed under the action of the derivative e_a. If we wish to consider irreducible spinors we must choose ψ in $\mathbf{C}^n$ and define the derivative

$$e_a\psi = m\lambda_a\psi.$$

If f is an element of $\mathcal{A}$, one sees that in fact

$$e_a(f\psi) = e_a(f)\psi + fe_a(\psi). \tag{4.11}$$

So this definition yields a proper derivation.

We could also have assumed that f transforms under the adjoint representation, that it is an isovector:

$$f \to g^{-1}fg. \tag{4.10}$$

In this case the covariant derivative would be

$$D_i = e_i + ad\omega_i, \tag{4.12}$$

and in the absence of all external fields, in the symmetric phase, it would become

$$D_a \equiv 0, \qquad D_\alpha = e_\alpha.$$

This equation means that the vertical component of the covariant derivative with respect to the flat connection θ of any scalar isovector f vanishes identically. The scalar-mode masses now vanish in the symmetric phase. The minimally-coupled internal laplacian is the zero operator; all of its eigenvalues vanish. Here again the masses remain as given by equation (4.6) in the broken phase.

5. Kaluza-Klein Theory We shall develop here the point of view that what we have done in fact in the previous sections is to generalize the original idea of Kaluza and Klein by replacing a point in space-time by an internal structure which is not a manifold nor even a topological space. We recall the definition of the manifold on which Kaluza-Klein theory is usually based. Locally it is of the form

$$\begin{array}{c} R^4 \times F \\ \downarrow \\ R^4 \end{array} \tag{5.1}$$

where F is a manifold. We have therefore

$$0 \to \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}(F) \tag{5.2}$$

where $\mathcal{C}$ designates an algebra of complex-valued functions, large enough to separate points. We have set $\mathcal{C}(R^4) = \mathcal{C}$ and we have supposed that the algebra of a product manifold can be identified with the product of the algebras of the individual factors. Kaluza-Klein theory in the usual sense can be described equally well by referring to (5.1) or to (5.2). The internal structure is described by the manifold F or by the algebra of functions $\mathcal{C}(F)$.

We can generalize Kaluza-Klein theory by replacing $\mathcal{C}(F)$ by an associative algebra which is not necessarily an algebra of functions. We can then no longer refer to the diagram (5.1) as there is no internal manifold F. This is in fact what we have done. We have chosen M_n to describe the internal structure. This choice has several advantages which we have mentioned, the most important being related to the fact that the space of derivations of M_n is of finite dimension. It follows in particular as we have already noted in the previous section that the internal laplacian has a finite spectrum and that the total momentum space remains of dimension 4. This is important for the renormalizability of the theory. A second advantage of interest to us in this section is that on the space of derivations of the total algebra $\mathcal{A}$ we can still define a linear connection although $\mathcal{A}$ is non-commutative. We wish to show this here and remark that this linear connection can be constructed in the usual Kaluza-Klein fashion from an ordinary linear connection on the base $\Omega^1(\mathcal{C})$ and a connection on $\mathcal{A}$.

Consider the imbedding

$$0 \to \mathcal{C} \to \mathcal{A} \tag{5.3}$$

given by $i : f \mapsto f \otimes 1$ and the corresponding imbedding of $\Omega^1(\mathcal{C})$ into $\Omega^1(\mathcal{A})$. The imbedding i respects the algebra structure and corresponds to the projection of a bundle onto a base space. This constitutes the first step in the Kaluza-Klein program, the definition of the internal structure. The next step is the construction of a linear connection on $\Omega^1(\mathcal{A})$ which is metric with respect to (2.8).

The gauge group $\mathcal{G}$ introduced in Section 3 acts on $\mathcal{A}$ and the ordinary coordinate transformations act on $\mathcal{C}$ and therefore also on $\mathcal{A}$. There is no connection nor correlation between these two actions. It is possible however to fix this relation [6]. For example consider only the Lorentz transformations acting on $\mathcal{C}$. We could require that the algebra $\mathcal{A}$ be a representation of the Lorentz group, that is, that $\mathcal{G}$ be also replaced by the Lorentz group. This requirement would lead necessarily to algebras with an infinite number of generators. The internal structure instead of representing internal-symmetry degrees of freedom, would represent the spin structure.

In Section 3 we studied the properties of the general connection (3.2). We wish to use ω and a linear connection on $\Omega^1(\mathcal{C})$ to construct a linear connection on $\Omega^1(\mathcal{A})$. To simplify the calculations we shall here suppose also that

$$\omega_a^0 = 0, \qquad A_\alpha^0 = 0, \tag{5.4}$$

and that $g \in \mathcal{SU}_n$, the local SU_n gauge transformations.

First we restrict our considerations to that special class of connections whose vertical component ω_V, in Ω_V^1, is equal to the canonical 1-form θ:

$$\omega_V = \theta. \tag{5.5}$$

That is from (3.2) we suppose that the Higgs field vanishes. We are then in the symmetric phase and the gauge bosons have vanishing masses. Since we wish to unify these bosons with the massless gravitons, it is understandable that this condition plays a special role. We have then also as a consequence

$$\Omega = F. \tag{5.6}$$

The curvature of the connection ω is constructed using the exterior derivative d which acts on the coefficients ω^a as well as on the basis λ_a. We wish to introduce an effective exterior derivative which acts only on ω^a but which includes the action of d on λ_a. Set

$$(\tilde{d}\omega^a)\lambda_a = d(\omega^a\lambda_a). \tag{5.7}$$

This can be rewritten as

$$\tilde{d}\omega^a = d\omega^a + m\,C^a{}_{bc}\,\omega^b \wedge \theta^c.$$

Define $\tilde{\theta}^a$ by the equation

$$\omega = -m\lambda_a\tilde{\theta}^a$$

and set

$$\tilde{\theta}^i = (\theta^\alpha, \tilde{\theta}^a).$$

Let $\omega^\alpha{}_\beta$ now be a linear connection on $\Omega^1(\mathcal{C})$, an $so(3,1)$-valued 1-form satisfying the structure equations

$$\begin{aligned} &d\theta^\alpha + \omega^\alpha{}_\beta \wedge \theta^\beta = 0, \\ &d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta = \Omega^\alpha{}_\beta. \end{aligned} \tag{5.8}$$

We must construct an $so(n^2+2,1)$-valued 1-form $\tilde{\omega}^i{}_j$ on $\Omega^1(\mathcal{A})$ satisfying the first structure equation

$$\tilde{d}\tilde{\theta}^i + \tilde{\omega}^i{}_j \wedge \tilde{\theta}^j = 0. \tag{5.9}$$

Because of (5.6) this construction is as usual. One can see that the equations (5.9) are satisfied if $\tilde{\omega}^i{}_j$ is given by the equations

$$\begin{aligned} \tilde{\omega}^\alpha{}_\beta &= \omega^\alpha{}_\beta - \frac{\kappa^2}{2}F_a{}^\alpha{}_\beta\,\omega^a, \\ \tilde{\omega}^a{}_b &= -\frac{1}{2}C^a{}_{bc}\,\omega^c, \\ \tilde{\omega}^a{}_\alpha &= -\frac{\kappa}{2}F^a{}_{\alpha\beta}\,\theta^\beta, \\ \tilde{\omega}^\alpha{}_a &= \frac{\kappa}{2}F_a{}^\alpha{}_\beta\,\theta^\beta. \end{aligned} \tag{5.11}$$

Consider next a general $\mathcal{SU}_n$ connection with a general Higgs field. The equations (5.9) become

$$\tilde{d}\tilde{\theta}^i + \tilde{\omega}^i{}_j \wedge \tilde{\theta}^j = \tilde{\Theta}^i. \tag{5.12}$$

with a non-vanishing torsion form given by

$$\tilde{\Theta}^\alpha = 0, \qquad \lambda_a\tilde{\Theta}^a = -\kappa(D\phi - \phi^2). \tag{5.13}$$

The torsion of the linear connection depends on the Higgs field.

The last step in the Kaluza-Klein construction is to consider the second structure equations

$$\tilde{d}\tilde{\omega}^i{}_j + \tilde{\omega}^i{}_k \wedge \tilde{\omega}^k{}_j = \tilde{\Omega}^i{}_j, \tag{5.14}$$

and the equations of motion which follow from a suitable action. We shall not discuss this here. To within terms which vanish with the curvature the most general possible

lagrangian which yields a renormalizable theory in the non-gravitational sector is the lagrangian (3.13) above.

6. Nonabelian Gauge Models The lagrangian (3.13) is a generalization of the Yang-Mills-Higgs-Kibble lagrangian, with a more elaborate Higgs sector. Since we have replaced complex-valued functions by functions with values in the matrix algebra M_n it is to be expected that each ϕ_a takes its values in M_n. The most original part of the lagrangian (3.13) is the potental term $V(\phi)$ which comes from the curvature of the vertical part of the connection. It is not the most general gauge-invariant polynomial in the Higgs field which would be allowed and we have had to introduce an extended BRS operator to insure that the form is maintained during renormalization. We have already noted that besides $\phi_a = 0$ we have $\phi_a = m\,\lambda_a$ as a classical vacuum state and that the orbit of this latter state under the action of the gauge group can be identified with SU_n. Since $\pi_2(SU_n) = 0$ there are no topologically stable soliton solutions.

Consider the case $n = 2$ and compare the above model with that of the standard model for the electroweak interactions. We have considered U_n gauge bosons. In anticipation of a parity-breaking mechanism we could equally well have considered the chiral $U_n \times U_n$ gauge bosons. Everything we have done could be repeated by replacing the matrix algebra M_n by the product algebra $M_n \times M_n$ and the connection ω by the sum $\omega^{(+)} + \omega^{(-)}$ of the right and left helicity components. The Higgs potential has then four stable minima. We could suppose that nature has chosen that minimum which corresponds to the $\phi^{(+)}$ in the symmetric phase and the $\phi^{(-)}$ in the broken phase. The massless non-abelian bosons would be confined and not appear as asymptotic states. We would then have as physical states two massless U_1 gauge bosons, one of which could be identified with the photon and $n^2 - 1$ massive SU_n gauge bosons which could be coupled only to left-handed fermions. There is of course no analogue of the Weinberg angle. The massive gauge bosons are neutral and they have all equal masses.

In the case $n = 3$ we would have a model for gluons plus a spurious U_1 abelian gauge potential provided we suppose that we are in the symmetric phase. The major difference with the standard model for the strong interactions lies in the fact that here the symmetric phase is a stable phase. The Higgs bosons are coloured and would not appear as asymptotic states.

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